Spark under 2-D Fourier Sampling

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***Abstract*—We consider the spark of submatrices of 2D-DFT matrices obtained by removing certain rows and relate it to the spark of associated 1D-DFT submatrices. A matrix has spark m if its smallest number of linearly dependent columns equals m. To recover an arbitrary k-sparse vector, the spark of an observation matrix must exceed $2k$. We consider how to choose the rows of the 2D-DFT matrix so that it is full spark, i.e. its spark equals one more than its row dimension. We consider submatrices resulting from two sets of sampling patterns in frequency space: On a straight line and on a rectangular grid. We show that in the latter case full spark is rarely obtainable, though vectors with certain sparsity patterns can still be recovered. In the former case we provide a necessary and sufficient condition for full spark, and show that lines with integer slopes cannot attain it.**

***Index Terms*—Coprime sensing, full spark, compressed sensing, two dimensional, Fourier Sampling**

I. Introduction

This paper derives conditions under which submatrices of a two dimensional DFT matrix, obtained by picking a subset of its rows, have full spark. The spark of a matrix $\boldsymbol{A} \in \mathbb{C}^{n\times m}$ is the smallest number of its columns that are linearly dependent. Such a matrix $\boldsymbol{A}$ has full spark if spark$(\boldsymbol{A}) = n+1$. The application rests in the recovery of sparse signals from their under-sampled linear measurements. Referred to as compressed sensing (CS), this approach has many applications, including MRI [2]–[4] and synthetic aperture radar [5]–[7].

In the general CS setting, the measurement process is modeled as

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x},$$

where $\boldsymbol{x} \in \mathbb{C}^m$ is the sparse signal vector to be recovered from the observations $\boldsymbol{y} \in \mathbb{C}^n$. We say that a vector $\boldsymbol{x}$ is k-sparse if it has at most k nonzero elements.

The *observation matrix* $\boldsymbol{A}$, is generally fat i.e. $n < m$. The condition spark$(\boldsymbol{A}) > 2k$, ensures that the unique recovery of a k-sparse $\boldsymbol{x}$ is possible from $\boldsymbol{y}$. [8], [9]. Violation of this condition implies there are at least two distinct k-spark vectors $\boldsymbol{z}$ and $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{z}$. The seminal work of Candes and Tao and its extensions show that further restrictions on $\boldsymbol{A}$ permit the recovery of $\boldsymbol{x}$ through convex ℓ_1 optimization [8], [10], [11].

As the acquisition of each measurement comes with a penalty (e.g. acquisition time in MRI), it is beneficial to recover $\boldsymbol{x}$ with as few measurements as possible. Thus it is desirable that one design a measurement matrix $\boldsymbol{A}$ with the highest possible spark. Computing the spark of a large matrix is clearly intractable. Thus early work on assessing the spark of a matrix relied on certain lower bounds involving the notion of mutual coherence between the columns of $\boldsymbol{A}$, [8], [12]–[15]. As explained in Section II and Section III, in many imaging applications like MRI and synthetic aperture radar, the observation matrices are certain rows of 2-D DFT matrices, [18]. This paper provides some initial results on the spark of such submatrices.

We note that the study of the spark of even 1-D DFT matrices is embryonic. Most advanced results are in [16] and in our own work in [17]. Motivated by developments in *coprime sampling*, [1], [17] in particular ties full spark submatrices of $N\times N$, 1-D DFT matrices to certain coprimeness conditions. We demonstrate in this paper that the spark of submatrices of 2-D DFT matrices obtained by certain specific types of sampling in frequency space (known as k-space) are related to the spark of associated 1-D DFT matrices. The sampling patterns considered are (i) on a line in k-space and (ii) on a rectangular grid.

Section III defines the sampling patterns we consider. Section IV recounts the results of [16] and a generalization of a result from [17]. Section V characterizes lines on k-space that result in full spark. One important conclusion is that while lines with rational slopes may lead to full spark observation matrices, lines with integer slopes never do. Section VI considers rectangular grids and shows that unless all rows of the 2-D DFT matrix are retained, full spark is impossible. It however, shows that such a matrix is still useful for the recovery of vectors with certain sparsity patterns. Section VII is the conclusion.

II. Background

The acquisition scheme in imaging schemes (e.g MRI, synthetic aperture radar) can be modeled as the 2-D Fourier samples of the continuous domain support limited object [18]:

$$\hat{\rho}[i] = \int_{\Omega} f(\boldsymbol{r}) \exp\left(-j2\pi \boldsymbol{k}_i^{\top}\boldsymbol{r}\right) d\boldsymbol{r};\ \boldsymbol{k}_i \in \mathbb{R}^2; i = 1,..,N. \tag{II.1}$$

Here, $\Omega \subseteq [0,1]\times[0,1]$ is the support of the object and $\boldsymbol{k}_i$ is the i^{th} frequency point at which the acquisition is performed. The acquisition time is directly proportional to the total number of Fourier samples, specified by N. Since the recovery of a continuous domain signal $f(\boldsymbol{r})$ from finite measurements is ill-posed, a common practice in MRI is to model the continuous 2-D signal using a shift invariant

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representation [18]

$$f(\boldsymbol{r})=\sum_{\boldsymbol{m}}\mathbf{c}[\boldsymbol{m}]\ \varphi\left(\frac{\boldsymbol{r}}{N}-\boldsymbol{m}\right). \tag{II.2}$$

Here, φ are the voxel basis functions and N is the sampling step. Note that higher values of N translate to higher resolution image models. Substituting the model in (II.1), one obtains, for suitable $\hat{\varphi}(\boldsymbol{k}_i)$, [18]

$$\begin{aligned}\hat{\rho}[i] &= \sum_{\boldsymbol{m}} c_{\boldsymbol{m}}\int_{\Omega}\varphi\left(\frac{\boldsymbol{r}}{N}-\boldsymbol{m}\right)exp\left(-j2\pi\boldsymbol{k}_i^{\top}\boldsymbol{r}\right)d\boldsymbol{r};\\ &\quad \boldsymbol{k}_i\in\mathbb{R}^2; i=1,..,N\\ &= \hat{\varphi}(\boldsymbol{k}_i)\sum_{\boldsymbol{m}} c_{\boldsymbol{m}}\exp\left(-j2\pi\boldsymbol{k}_i^{\top}\boldsymbol{m}\right);\\ &\quad \boldsymbol{k}_i\in\mathbb{R}^2; i=1,..,N\end{aligned}$$

If the voxel basis functions are sinc functions, $\hat{\varphi}(\boldsymbol{k}_i)=1$ within the bandwidth and is zero otherwise. If other models (e.g spline functions are used), the magnitude of $\hat{\varphi}(\boldsymbol{k}_i)$ and $\hat{\rho}[i]$ will decrease rapidly with increasing $\boldsymbol{k}$.

In MR imaging applications, the acquisition of the uniform Fourier samples on lines in k-space is efficient. Specifically, the frequency encoding gradients along x and y directions are kept constant, while reading out the data at uniform intervals; the slope of the line is controlled by the ratio of the frequency encoding gradients [18]. This allows us to acquire a line of k-space data in a single excitation. Phase encoding gradients can be used to shift the line from the origin. If Cartesian sampling is used, the data is acquired on equispaced lines in k-space; N excitations are required to Nyquist sample the image. An alternative is radial sampling [18], where the slope of the lines are varied from excitation to excitation.

III. Notation and Preliminaries

We define $\boldsymbol{W}_N$ to be the $N\times N$ 1-D, DFT matrix with row and column indices taking values from the set $\mathbb{Z}_N=\{0,1,\cdots,N-1\}$. The il-th element of $\boldsymbol{W}_N$ is $e^{\frac{j2\pi il}{N}}$. On the other hand, the 2-D DFT matrix has rows and columns indexed by 2-vectors $\boldsymbol{k}$ and $\boldsymbol{n}$. There is one row per vector $\boldsymbol{k}$, and one column per vector $\boldsymbol{n}$. Throughout we assume that

$$\boldsymbol{n}\in\mathbb{Z}_N^2, \tag{III.1}$$

i.e. it has precisely N^2 columns. For a given $\boldsymbol{k}$, the corresponding row of the DFT-matrix will be denoted

$$\boldsymbol{W}^{(2)}(\boldsymbol{k},\boldsymbol{n})=\exp\left\{j\frac{2\pi\boldsymbol{k}^{\top}\boldsymbol{n}}{N}\right\},\boldsymbol{n}\in\mathbb{Z}_N^2. \tag{III.2}$$

The conventional 2-D DFT matrix [20] assumes that $\boldsymbol{k}\in\mathbb{Z}_N^2$, i.e. under (III.2) the matrix is in $\mathbb{C}^{N^2\times N^2}$.

We are interested in the spark of a 2-D DFT matrices that have more columns than rows. The following special cases are of interest here.

- A rectangular subgrid: For some subsets $S_i\subset\mathbb{Z}_N$ and $\boldsymbol{D}=\text{diag}\ \{d_1,d_2\},\ d_i\in\mathbb{Z}_+$
$$\boldsymbol{W}^{(2)}(\boldsymbol{k},\boldsymbol{n})=\exp\left\{j\frac{2\pi\boldsymbol{k}^{\top}\boldsymbol{D}\boldsymbol{n}}{N}\right\},\boldsymbol{k}\in S_1\times S_2,\ \boldsymbol{n}\in\mathbb{Z}_N^2. \tag{III.3}$$
Here $\mathbb{Z}_+$ is the set of positive integers. We will denote the matrix thus formed $\boldsymbol{W}^{(2)}(S,\boldsymbol{D})$.
- A line with rational slope: Under (III.2), and L and M integers $\boldsymbol{k}=[k_1,k_2]^{\top}$ are on a line:
$$K(L,M,S)=\{\boldsymbol{k}|\,Lk_1-Mk_2=0,k_1\in S\subset\mathbb{Z}_N\} \tag{III.4}$$
We will denote matrix thus formed by $\boldsymbol{W}^{(2)}(K(L,M,S))$. Evidently $K(L,M,S)$ has a slope L/M in k-space.

Unlike the rectangular subgrid where each element of $\boldsymbol{k}$ takes integer values, in (III.4) the second element of $\boldsymbol{k}$ may be rational. Sampling in k-space along a line such as (III.4) is quite common in MRI applications.

Define:

$$\boldsymbol{w}_N(l)=\left[1\quad e^{\frac{j2\pi l}{N}}\quad\cdots\quad e^{\frac{j2\pi l(N-1)}{N}}\right]. \tag{III.5}$$

Define $\boldsymbol{e}_1=[1,0]^{\top}$ and $\boldsymbol{e}_2=[0,1]^{\top}$. Then for suitable permutation matrices $\boldsymbol{P}_i\in\mathbb{R}^{N^2\times N^2}$ the row $\boldsymbol{W}^{(2)}(\boldsymbol{k},\boldsymbol{n})$ in (III.2) is expressible as

$$\begin{aligned}\boldsymbol{W}^{(2)}(\boldsymbol{k},\boldsymbol{n}) &= \boldsymbol{w}_N(\boldsymbol{k}^{\top}\boldsymbol{e}_2)\otimes\boldsymbol{w}_N(\boldsymbol{k}^{\top}\boldsymbol{e}_1)\boldsymbol{P}_1\\ &= \boldsymbol{w}_N(\boldsymbol{k}^{\top}\boldsymbol{e}_1)\otimes\boldsymbol{w}_N(\boldsymbol{k}^{\top}\boldsymbol{e}_2)\boldsymbol{P}_2\end{aligned}$$

Define $\boldsymbol{W}_N(S)$ to comprise the submatrix of $\boldsymbol{W}_N$ formed by the rows indexed by $S\in\mathbb{Z}_N$. Then evidently, (III.6) leads to the following expression: For permutation matrices $\boldsymbol{P}_i,\boldsymbol{Q}_i$,

$$\begin{aligned}\boldsymbol{W}^{(2)}(S,\boldsymbol{D}) &= \boldsymbol{P}_1\boldsymbol{W}_N(d_1S)\otimes\boldsymbol{W}_N(d_2S)\boldsymbol{Q}_1\\ &= \boldsymbol{P}_2\boldsymbol{W}_N(d_2S)\otimes\boldsymbol{W}_N(d_1S)\boldsymbol{Q}_2\end{aligned} \tag{III.6}$$

Here for a scalar d, if $S=\{s_1,\cdots,s_n\}$ then $dS=\{ds_1,\cdots,ds_n\}$.

IV. Spark of submatrices of $\boldsymbol{W}_N$

We observe that even the results on the spark of submatrices $\boldsymbol{W}_N(S)$ defined in the previous section are few, [16], [17]. Here we recount a few that are used in later sections. We also provide a new result that to our knowledge is new.

Suppose for some i and l, $S=\{i,i+1,\cdots,i+l-1\}$ i.e. S contains consecutive elements of $\mathbb{Z}_N$. Define

$$z_n=e^{j\frac{2\pi n}{N}} \tag{IV.1}$$

Then the matrix comprising any l columns of $\boldsymbol{W}_N(S)$, indexed by the integers $i_1,\cdots,i_l$ can be expressed as

$$\begin{bmatrix}1 & 1 & \cdots & 1\\ z_{i_1} & z_{i_2} & \cdots & z_{i_l}\\ \vdots & \vdots & \vdots & \vdots\\ z_{i_1}^{l-1} & z_{i_2}^{l-1} & \cdots & z_{i_l}^{l-1}\end{bmatrix}\text{diag}\ \{z_{i_1}^{i},z_{i_2}^{i},\cdots,z_{i_l}^{i}\}.$$

and being a product of a Vandermonde and a nonsingular diagonal matrix, is thus nosingular for distinct z_{i_n}. Consequently, such a $\boldsymbol{W}_N(S)$ has full spark. Examples where nonconsecutive rows cause spark to be lost can be found in [16] and [17]. An exception is provided by a classical result that can be traced back to Chebotarëv, (see [19]).

Theorem 4.1: Suppose N is prime. Then for all $S \subset \mathbb{Z}_N$, $\boldsymbol{W}_N(S)$ has full spark.

The results of [16] below provide a way to generate additional sets $\bar{S}$, beyond those comprising consecutive integers, for which $\boldsymbol{W}_N(\bar{S})$ has full spark.

Theorem 4.2: Suppose for some $S \subset \mathbb{Z}_N$, $\boldsymbol{W}_N(S)$ has full spark. Then so does:

(i) $\boldsymbol{W}_N((S+i) \mod N)$ for all $i \in \mathbb{Z}_N$.
(ii) $\boldsymbol{W}_N(MS)$ for all M that is coprime with N.
(iii) $\boldsymbol{W}_N(\mathbb{Z}_N \setminus S, N)$.

We next provide Theorem 4.3 that generalizes a result in [17]. It requires the following Lemma from [20].

Lemma 4.1: Consider integers $1 \leq M < N$, M and N not coprime. Then there exists $1 \leq n < N$ such that N divides Mn.

Using this lemma we now prove the following theorem.

Theorem 4.3: For integer $1 < M < N$, $S \in \mathbb{Z}_N$, with $|S| > 1$ suppose $\boldsymbol{W}(S)$ has full spark. Then $\boldsymbol{W}_N(MS)$ has full spark if and only if M and N are coprime. Otherwise spark$(\boldsymbol{W}_N(MS)) = 2$.

Proof: Sufficiency follows from Theorem 4.2. Now suppose M and N are not coprime. Then from Lemma 4.1 there is an $1 \leq n < N$ such that N divides Mn. Then the column of $\boldsymbol{W}(MS)$ indexed by n is a vector of ones. Further the first column is also a vector of ones. The result follows. ■

The corresponding result in [17] assumes that S contains consecutive integers.

V. Spark on a line with rational slope

We now consider the special case when the k-space samples lie on a line with possible noninteger slopes. Specifically, with $K(L, M, S)$ as in (III.4) and L and M integers we consider submatrices whose rows are indexed by $\boldsymbol{k} \in K(L, M, S)$ and $\boldsymbol{n} \in \mathbb{Z}_N^2$. Observe that with $\boldsymbol{k} = [k_1, k_2]^\top$, under (III.4) one has:

$$\begin{aligned} \boldsymbol{W}^{(2)}(\boldsymbol{k}, \boldsymbol{n}) &= \exp\left(j\frac{2\pi k_1}{NM}(Mn_1 + Ln_2)\right) \\ &= \exp\left(j\frac{2\pi k_1 [M, L]\boldsymbol{n}}{NM}\right). \end{aligned} \quad \text{(V.1)}$$

For the moment we make the following simplifying assumption, which we will relax later.

Assumption 5.1: The integers M and L in (V.1) are coprime.

Under this assumption we have the following lemma.

Lemma 5.1: Suppose under Assumption 5.1 and $k = 1$, there exist distinct $\boldsymbol{n}_i \in \mathbb{Z}_N^2$ such that:

$$k[M, L]\boldsymbol{n}_1 \mod MN = k[M, L]\boldsymbol{n}_2 \mod MN \quad \text{(V.2)}$$

Then:

(i) $M < N$, and
(ii) (V.2) holds for all $k \in \mathbb{Z}$.

Further, if $M < N$ then there exist distinct $\boldsymbol{n}_i \in \mathbb{Z}_N^2$ such that (V.2) holds for $k = 1$.

Proof: Suppose (V.2) holds with $k = 1$. Then there exists an integer $r \in \mathbb{Z}_{MN}$, and $q_i \in \mathbb{Z}$ such that for $i \in \{1, 2\}$

$$[M, L]\boldsymbol{n}_i = q_i MN + r. \quad \text{(V.3)}$$

Call n_{ij} the j-th element of $\boldsymbol{n}_i$. Then (V.3) we have:

$$M(n_{11} - n_{21}) + L(n_{12} - n_{22}) = (q_1 - q_2)MN.$$

First suppose that $n_{12} = n_{22}$. As $\boldsymbol{n}_1 \neq \boldsymbol{n}_2$ this must mean that N divides $|n_{11} - n_{21}|$. As $|n_{11} - n_{21}| < N$ this is impossible. Thus $n_{12} \neq n_{22}$. Further,

$$M\left[(q_1 - q_2)N + n_{21} - n_{11}\right] = L(n_{12} - n_{22}).$$

As M and L are coprime this means that M divides $|n_{12} - n_{22}|$ implying (i). Further, under (V.2), for all $k \in \mathbb{Z}$ and $i \in \{1, 2\}$,

$$k[L, M]\boldsymbol{n}_i = kq_i MN + kr,$$

i.e.

$$k[L, M]\boldsymbol{n}_i \mod NM = kr \mod NM$$

and (ii) holds. Finally, suppose $M < N$. Then the set

$$\left\{[L, M]\boldsymbol{n} \mod NM \,\middle|\, \boldsymbol{n} \in \mathbb{Z}_N^2\right\}$$

has at most NM elements, though $\mathbb{Z}_N^2$ has N^2 elements. As $MN < N^2$, there must be two distinct $\boldsymbol{n}_i \in \mathbb{Z}_N^2$ such that (V.2) holds for $k = 1$. ■

The next lemma establishes a sufficient condition for a full spark $\boldsymbol{W}^{(2)}(k(L, M, S))$.

Lemma 5.2: Suppose Assumption 5.1 holds, $M \geq N$ and for some $N > l \geq 2$, $S = \mathbb{Z}_l$. Then $\boldsymbol{W}^{(2)}(K(L, M, S))$ has full spark. Further if $M < N$, then $\boldsymbol{W}^{(2)}(K(L, M, S))$ has spark 2.

Proof: The column indexed by $\boldsymbol{n}_i \in \mathbb{Z}_N^2$ is

$$[1, x_i, \cdots x_i^{l-1}]^\top$$

where

$$x_i = \exp\left(j\frac{2\pi[M, L]\boldsymbol{n}_i}{NM}\right). \quad \text{(V.4)}$$

Thus, the Vandermonde structure ensures full spark unless for distinct $\boldsymbol{n}_i \in \mathbb{Z}_N^2$, (V.2) holds. Lemma 5.1 shows that this is false if $M \geq N$. On the other hand if $M < N$ there exist distinct $\boldsymbol{n}_1, \boldsymbol{n}_2 \in \mathbb{Z}_N^2$ such that $x_1 = x_2$ in (V.4) and $\boldsymbol{W}^{(2)}(K(L, M, S))$ has spark 2. ■

In fact using, Theorem 4.2 and 4.3 a stronger result can be obtained.

Theorem 5.1: Suppose for some integer N, the set S is such that $\boldsymbol{W}_{MN}(S)$ has full spark and Assumption 5.1 holds. Then for given integer p, $\boldsymbol{W}^{(2)}(K(L, M, pS))$ has full spark iff p and MN are coprime and $M \geq N$.

Proof: The proof follows by considerations in the proof of Lemma 5.2, Theorem 4.2 and Theorem 4.3 and the fact that $\boldsymbol{W}^{(2)}(K(L, M, pS))$ is a submatrix of and has the same number of rows as $\boldsymbol{W}_{MN}(pS)$. ∎

In summary, this result has two features. First it translates the spark condition for a 2-D DFT matrix in to one for a 1-D DFT matrix. Second it demands that for a line of slope L/M in k-space, with L and M coprime, M cannot be smaller than N. This precludes integer slopes, or for that matter a 45^o line. Though of course one can sample on lines that are arbitrarily close in their slope to any given line.

To understand how Assumption 5.1 can be relaxed suppose M and L have greatest common divisor (gcd) p. Define

$$\bar{M} = \frac{M}{p} \text{ and } \bar{L} = \frac{L}{p} \tag{V.5}$$

Then $\bar{M}$ and $\bar{L}$ are coprime and

$$\frac{2\pi k_1}{NM}(Mn_1 + Ln_2) = \frac{2\pi k_1}{N\bar{M}}(\bar{M}n_1 + \bar{L}n_2).$$

Then all one needs to do is to replace M by $\bar{M}$ in both Lemma 5.2 and Theorem 5.1.

VI. Spark on a rectangular subgrid

We now turn to the setting of rectangular sampling in k-space and first provide a seemingly negative result. Later in the section we show that in fact there are vectors with particular sparsity patterns that can be recovered through rectangular sampling.

First observe from (III.6), that to within permutations of rows and columns the matrix in (III.3) is the Kronecker product of submatrices of a 1-D DFT matrix. In general, barring pathologies the rank of a Kronencker product exceeds that of the factors. Theorem 6.1 below shows that this is not the case for spark of Kronecker products.

Theorem 6.1: Suppose $\boldsymbol{A} \in \mathbb{C}^{n_A \times m_A}$ and $\boldsymbol{B} \in \mathbb{C}^{n_B \times m_B}$. Then the following hold.

(i) If $n_B < m_B$ then,

$$\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) \le \text{spark}(\boldsymbol{B}).$$

(ii) If $n_A < m_A$ then,

$$\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) \le \text{spark}(\boldsymbol{A}).$$

(iii) If $n_B < m_B$, and $\text{spark}(\boldsymbol{A}) \ge \text{spark}(\boldsymbol{B})$

$$\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) = \text{spark}(\boldsymbol{B}).$$

Proof: To prove (i) suppose $\text{spark}(\boldsymbol{B}) = p \le n_B$. Observe that the first block column of $\boldsymbol{A} \otimes \boldsymbol{B}$ is given by

$$\begin{bmatrix} a_{11}B \\ \vdots \\ a_{m1}B \end{bmatrix}.$$

Then there exists a p-sparse $\boldsymbol{x} \neq 0$ such that $\boldsymbol{Bx} = 0$. Then

$$\begin{bmatrix} a_{11}B \\ \vdots \\ a_{m1}B \end{bmatrix} \boldsymbol{x} = 0.$$

Thus (i) follows. The proof of (ii) follows from (i), the fact that for any pair of permutation matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$

$$\text{spark}\,(\boldsymbol{P}(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{Q}) = \text{spark}\,(\boldsymbol{A} \otimes \boldsymbol{B})$$

and the fact that there exist permutation matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{P}(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{Q} = \boldsymbol{B} \otimes \boldsymbol{A}. \tag{VI.1}$$

Now suppose $\text{spark}(\boldsymbol{A}) \ge \text{spark}(\boldsymbol{B}) = p \le n_B$. If $p = 1$ then $\boldsymbol{B}$ has a zero column and so does $\boldsymbol{A} \otimes \boldsymbol{B}$. Thus indeed $\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) = 1 = p$. Now suppose $p > 1$ but $\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) = q < p$. Thus there exists a q-sparse vector $\boldsymbol{x}$ such that

$$(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{x} = 0.$$

Then there exist $M \le q$, vectors $\boldsymbol{x}_l \neq 0$, $l \in S_M$ each at most q-sparse, such that for all $i \in S_{n_A}$ there holds:

$$\sum_{l \in S_M} a_{il}\boldsymbol{B}\boldsymbol{x}_l = 0. \tag{VI.2}$$

Further as $\text{spark}(\boldsymbol{B}) > q$, for all $l \in S_M$, $\boldsymbol{B}\boldsymbol{x}_l \neq 0$ Thus there exits a nonzero M-sparse vector $\boldsymbol{y}$ such that

$$\boldsymbol{Ay} = 0.$$

Thus $\text{spark}(\boldsymbol{A}) \le M \le q < p = \text{spark}(\boldsymbol{B})$, establishing a contradiction. Thus $\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) \ge p$. From (i) this means that $\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) = p$. ∎

Thus rectangular sampling in k-space does not offer benefits in terms of impoved spark. In fact in light of Theorem 4.3 should either d_1 or d_2 in (III.6) share a common factor with N, then spark of the 2-D DFT matrix will be just two.

Yet there are specialized settings where Kronecker products carry increased benefits. Specifically, Kronecker products exhibit certain advantages in recovery of vectors that are *jointly sparse*. Thus consider

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_m \end{bmatrix} \in \mathbb{C}^{n \times m}. \tag{VI.3}$$

We call the set of vectors $\boldsymbol{x}_i$ jointly k-sparse if $\boldsymbol{X}$ has at most k nonzero rows. Now, with $\boldsymbol{A} \in \mathbb{C}^{n_A \times m_A}$ and $\boldsymbol{B} \in \mathbb{C}^{n_B \times n}$, consider the observations:

$$\boldsymbol{y} = (\boldsymbol{A} \otimes \boldsymbol{B})\mathbf{vec}(\boldsymbol{X}), \tag{VI.4}$$

where

$$\mathbf{vec}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix}.$$

Observe $\mathbf{vec}(\boldsymbol{X})$ is mk-sparse. Thus, on the face of it unique recovery of $\boldsymbol{X}$ from $\boldsymbol{y}$ requires:

$$\text{spark}(\boldsymbol{A} \otimes \boldsymbol{B}) > 2mk. \tag{VI.5}$$

Theorem 6.1 would then suggest the requirement,

$$\min\{\text{spark}(\boldsymbol{A}), \text{spark}(\boldsymbol{B})\} > 2mk. \tag{VI.6}$$

We now argue that (VI.6) is in fact too conservative.

Indeed with

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_{n_A} \end{bmatrix},$$

(VI.4) reduces to:

$$\boldsymbol{y}_i = \boldsymbol{B}\left(\sum_{l=1}^{m} a_{il}\boldsymbol{x}_l\right), \ \forall i \in S_{n_A}. \qquad \text{(VI.7)}$$

As $\boldsymbol{X}$ can have at most k nonzero rows, for all $i \in S_{n_A}$, $\boldsymbol{q}_i = \sum_{l=1}^{m} a_{il}\boldsymbol{x}_l$ is k-sparse. Thus spark$(\boldsymbol{B}) > 2k$ suffices to uniquely determine all the $\boldsymbol{q}_i$, and in the process determining the joint support of the columns of $\boldsymbol{X}$. Finally, spark$(\boldsymbol{A}) > m$ suffices to extract the $\boldsymbol{x}_i$ from the $\boldsymbol{q}_i$. Thus far smaller values for spark$(\boldsymbol{A})$ and spark$(\boldsymbol{B})$ than that suggested by (VI.6) suffices to resolve this joint sparsity problem.

VII. Conclusion

We have considered the spark of submatrices of 2-D DFT matrices under two specific types of sampling in k-space. In both cases we show that spark is related to the spark of certain related 1-D DFT matrices. The first sampling pattern, we consider involves a single line in k-space and provide a necessary and sufficient conditions for full spark. One interesting conclusion is that lines with integer slopes preclude full spark. The second type of sampling is on rectangular grid. We show that full spark is in general impossible in such a setting. However, we also provide conditions under which such matrices can recover jointly sparse vectors.

An open problem of substantial benefit to MRI is as follows: Instead of sampling on a single line in k-space suppose we sampled over several lines. How to extend the results of Section V in such a setting? We are currently working towards resolving this problem, possibly through the aid of Theorem 5.1.

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