Recovery of noisy points on band-limited surfaces

Sunrita Poddar, Mathews Jacob
Many datasets of points lie on a surface

Hands dataset
http://web.mit.edu/cocosci/isomap/datasets.html

ADNI dataset
On the manifold structure of the space of brain images, Gerber et al
MR images parametrized by cardiac and respiratory phases

Recovery from few measurements

Subspace based models:
1. XD-GRASP: Golden-angle radial MRI with reconstruction of extra motion-state dimensions using compressed sensing. L. Feng et al MRM 2015
Outline

➢ Union of curves model

➢ Recovery of curves from samples

➢ Solving inverse problems using model

➢ Summary
Outline

➢ Union of curves model

➢ Recovery of curves from samples

➢ Solving inverse problems using model

➢ Summary
Model: Zero-level set of bandlimited function

\[ \{ \mathbf{r} \in \mathbb{R}^n | \psi(\mathbf{r}) = 0 \} \]

Zero-level set

\[ \psi(\mathbf{r}) = \sum_{\mathbf{k} \in \Lambda} c_k e^{j 2\pi \mathbf{k}^T \mathbf{r}} \]

Band-limited function

Fourier coefficients
Rich enough to capture complex surfaces

Curve complexity increases with increase in bandwidth

\[ \psi(x, y) = 0 \]

Moving to higher dimensions

\[ \psi(x, y, z) = 0 \]

7x9 coefficients 13x13 coefficients 5x5x5 coefficients
Aim of this work

➢ Problem 1: Recovery of curves from sampled points

➢ Problem 2: Recovery of points on the curve from corrupted measurements
Model property: filter coefficients annihilate feature matrix

For $N$ points $\{x_i\}, i = 1, \ldots, N$ on the curve: $\psi(x_i) = 0$

$$\sum_{k \in \Lambda} c_k e^{j 2\pi k^T x_i} = 0$$

$$c^T \begin{bmatrix} 
  e^{j 2\pi k_1^T x_i} \\
  e^{j 2\pi k_2^T x_i} \\
  \vdots \\
  e^{j 2\pi k_{|\Lambda|}^T x_i} 
\end{bmatrix} \phi(x_i) = 0 \quad \Rightarrow \quad c^T \begin{bmatrix} 
  \phi(x_1) \\
  \phi(x_2) \\
  \vdots \\
  \phi(x_N) 
\end{bmatrix} = 0$$

Filter $c$ annihilates Feature matrix
Outline

➢ Union of curves model

➢ Recovery of curves from samples

➢ Solving inverse problems using model

➢ Summary
Recovery of curve from samples

Model property: Filter $c$ annihilates Feature matrix $c^T \Phi(X) = 0$

Aim: Recover curve $\psi(r) = 0$ from samples $x_1, x_2, \ldots, x_N$

Step-1: Form feature matrix $\Phi(X)$

Step-2: Find null-space vector $c$

Step-3: Inverse Fourier transform gives polynomial $\psi(r)$

Step-4: Take zero level-set to get $\psi(r) = 0$
Proposition-1: Number of samples for perfect recovery

Polynomial $\psi(r)$ with Fourier support $\Lambda$

$N$ points lying on $\psi(r) = 0$

How many points required to recover curve uniquely?

Case-1

$\psi(r)$ is irreducible

$N \geq (K_1 + K_2)^2$ samples

Proof uses Bežout’s inequality

Case-2

$J$ irreducible factors

$\psi(r) = \psi_1(r)\psi_2(r) \ldots \psi_J(r)$

$N_j \geq (K_1 + K_2)(K_{1,j} + K_{2,j})$ samples on $j^{th}$ factor
Proposition-1: Number of samples for perfect recovery

Fourier co-efficients: 5x5 support

\[ \psi(x, y) \]

\[ \psi(x, y) = 0 \]

Recovery of curve from samples:

- 5 points
- 10 points
- 25 points
- 50 points
Proposition-1: Comparing to a degrees of freedom argument

\[ (K_1 + K_2)^2 \]

\[ K_1 K_2 = |\Lambda| \]

Are \(|\Lambda|\) points sufficient?

\[ \psi_1(x, y) \equiv \cos(2\pi(x + y)) \]

\[ \psi_1(x, y) = 0 \]

\[ \psi_2(x, y) \equiv \cos(2\pi(x - y)) \]

\[ \psi_2(x, y) = 0 \]

12 intersection points

\[ |\Lambda| = 9 \]

but > 12 samples for unique recovery
Proposition-2: Recovery using over-estimated filter

Minimal filter support
Required $\Lambda$ is unknown!

Filter support over-estimated in practice

$N$ points lying on $\psi(r) = 0$

$N \geq (L_1 + L_2)(K_1 + K_2)$

$\psi(r)$ is irreducible

Multiple solutions of form:

$\hat{\psi}(r) = \psi(r)\eta(r)$

for arbitrary $\eta(r)$

Common zeros of all solutions give $\psi(r) = 0$

Reducible polynomials: $N_j \geq (L_1 + L_2)(K_{1,j} + K_{2,j})$
Proposition-2: Recovery using over-estimated filter

Fourier co-efficients: 5x5 support

Recovered polynomials from 100 samples using 11x11 support

$\psi(x, y)$

$\psi(x, y) = 0$

Sum-of-squares combined
Outline

➢ Union of curves model

➢ Recovery of curves from samples

➢ Solving inverse problems using model

➢ Summary
Solving inverse problems using model

Problem: Recover points \( \{x_i\} \) from corrupted measurements:

\[ b_i = A_i(x_i) + \eta_i \]

Example: Denoising problem
\[ \{b_i\} \text{ lie near curve } \{r \in \mathbb{R}^n | \psi(r) = 0\} \]
\[ \psi(r) : \text{Bandlimited function} \]

Solution: Use model properties
Proposition-3: Rank of the feature matrix

Minimal filter support
With sufficient points $N$: Unique null-space vector $c$

$$\text{rank} (\Phi(X)) = |\Lambda| - 1$$

Over-estimated filter support
With sufficient points $N$:

$$\text{rank} (\Phi(X)) = |\Gamma| - |\Gamma : \Lambda|$$

Feature matrix is rank-deficient

Number of valid shifts of $\Lambda$ in $\Gamma$
Enforcing low rank feature matrix to solve inverse problems

\[ \min_X \| \mathbf{A}(X) - b \|^2 + \lambda \| \Phi(X) \|_* \]

Iterative reweighted least squares scheme

\[ X^{(n)} = \arg \min_X \| \mathbf{A}(X) - b \|^2 + \lambda \text{ trace}[\mathbf{K}(X)\mathbf{Q}^{(n-1)}] \]

\[ \mathbf{Q}^{(n)} = [\mathbf{K}(X^{(n)}) + \gamma^{(n)} \mathbf{I}]^{-\frac{1}{2}} \]

Gradient linearization

\[ X^{(n)} = \arg \min_X \| \mathbf{A}(X) - b \|^2 + \lambda \text{ trace}(X^T\mathbf{L}^{(n-1)}X) \]

where \[ \mathbf{L}^{(n-1)} = f(\mathbf{K}(X^{(n-1)}), \mathbf{Q}^{(n-1)}) \]

Introduced for Polynomial Kernels in “Algebraic Variety Models for High-Rank Matrix Completion”, G. Ongie et al MLR 2017
Denoising synthetic data

Circle

Tigerhawk logo

Noisy points

1\textsuperscript{st} iteration

50\textsuperscript{th} iteration
Relation between model and kernel low-rank methods

Feature matrix

\[ \Phi(X) = \begin{bmatrix} \phi(x_1) & \phi(x_2) & \ldots & \phi(x_N) \end{bmatrix} \]

Size: \(|\Gamma| \times N\)  \(|\Gamma| \) increases exponentially with number of dimensions

Computing the Gram matrix

\[ \mathcal{K}(X) = \Phi(X)^H \Phi(X) \quad \Rightarrow \quad \text{Kernel matrix} \]

Size: \(N \times N\)  Size independent of ambient dimension

Dirichlet kernel
Free-breathing cardiac MR reconstruction

“Dynamic MRI using SToRM”
S. Poddar et al, TMI 2016

\[
\min_{X} \| A(X) - b \|^2 + \lambda \text{Tr}(X^T L X)
\]
Improved estimation of Laplacian eigen vectors

- Exponential weights
- Thresholded exponential weights
- Proposed

➢ Depends on threshold
➢ Does not capture physiological signal

Respiratory motion
Cardiac motion
Approximation of image series using few basis functions

\[ X = U_r V_r \]

Only \( r \) basis images to be reconstructed

\( r = 30 \) basis functions

Exponential weights

Exp Weights + UV factorization

Proposed

Exponential weights

20 min

Slow

Exp Weights + UV factorization

2 min

Motion artefact

Proposed

2 min
Reconstructed free-breathing cardiac datasets

2-chamber view

4-chamber view

Short axis view

Short axis view

10 patients recruited at the University of Iowa Hospitals and Clinics
➢ Union of curves model

\[ \psi(r) \quad \Rightarrow \quad \{ r \in \mathbb{R}^n | \psi(r) = 0 \} \]

➢ Guarantees for recovery of curves from their samples

➢ Solving inverse problems using low-rank feature matrix

➢ Connection to kernels and graph Laplacian

Questions?