Recovery of noisy points on band-limited surfaces

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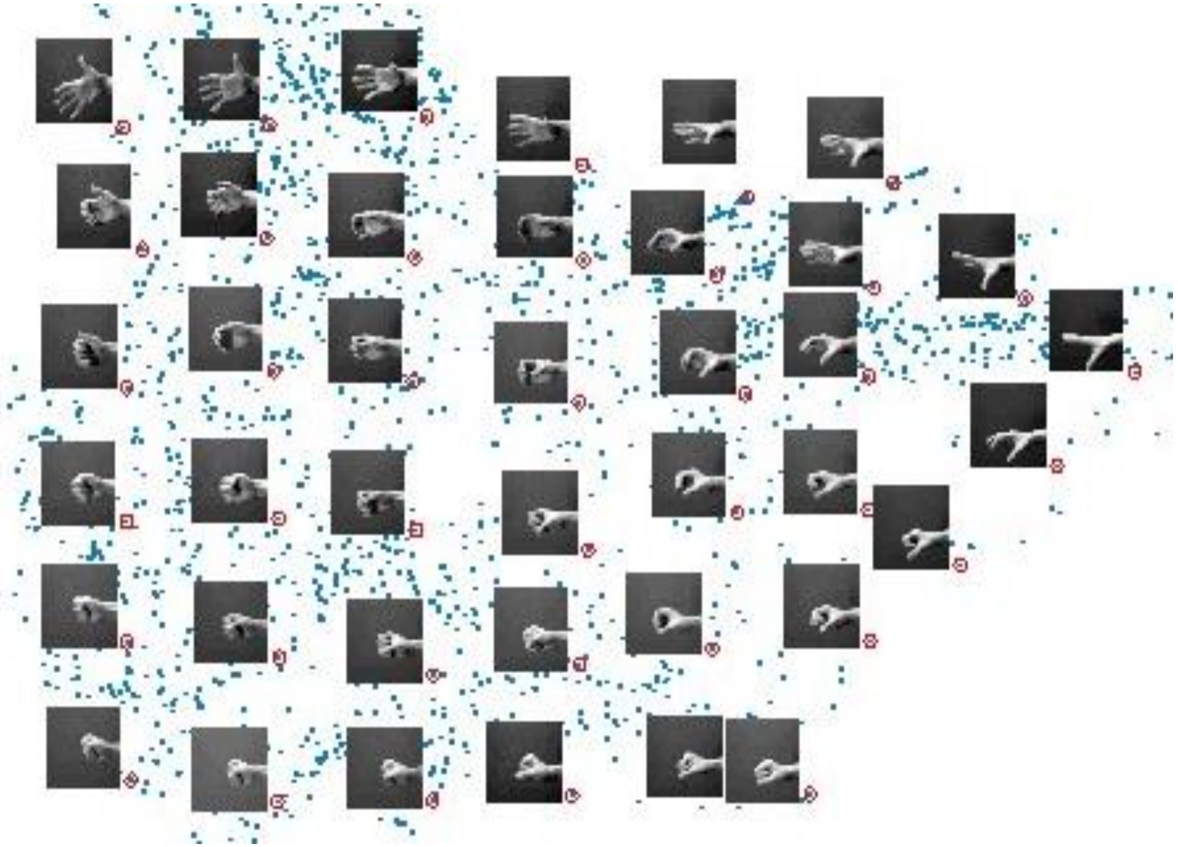
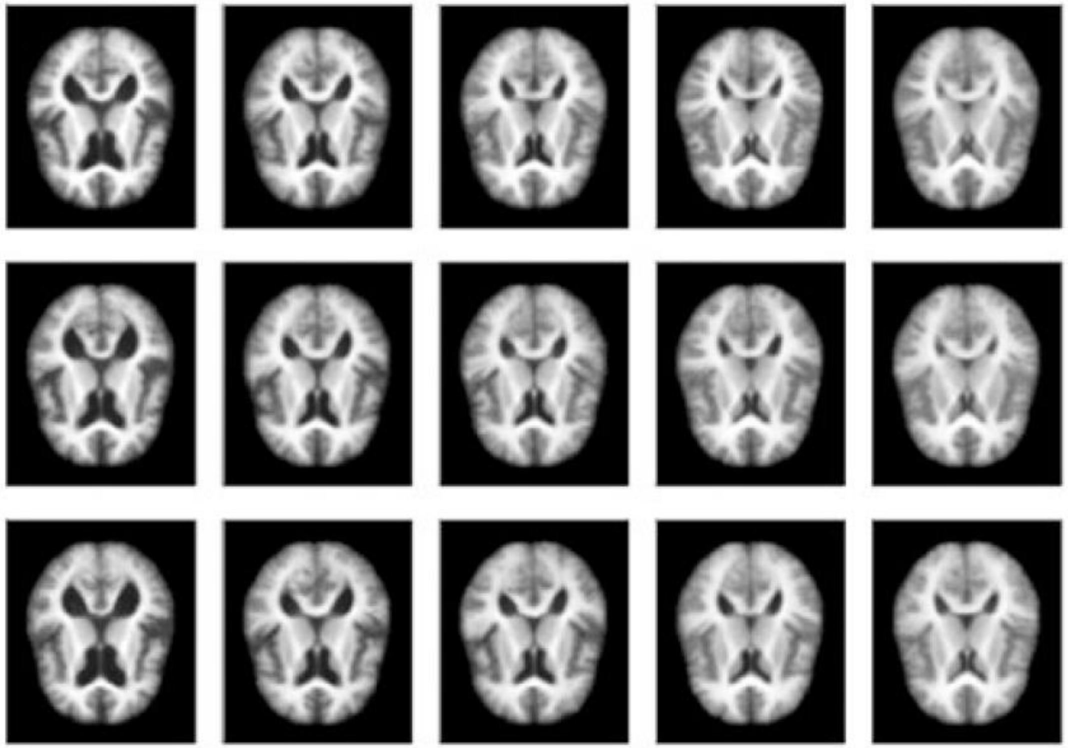
The University of Iowa

Many datasets of points lie on a surface



Hands dataset

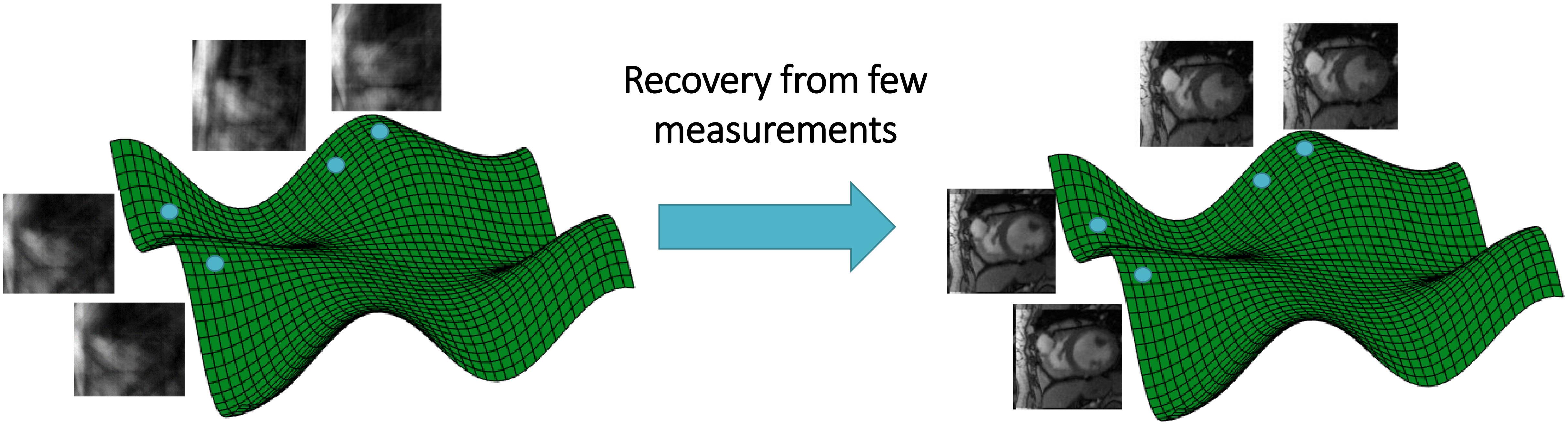
http://web.mit.edu/cocosci/isomap/datasets.html



ADNI dataset

On the manifold structure of the space of brain images, Gerber et al

MR images parametrized by cardiac and respiratory phases



Subspace based models:

1. XD-GRASP: Golden-angle radial MRI with reconstruction of extra motion-state dimensions using compressed sensing. L. Feng et al MRM 2015
2. Accelerated high-dimensional MR imaging with sparse sampling using low-rank tensors. J. He et al TMI 2016

Outline

- Union of curves model
- Recovery of curves from samples
- Solving inverse problems using model
- Summary

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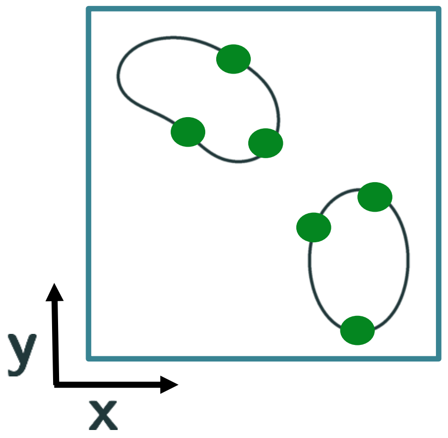
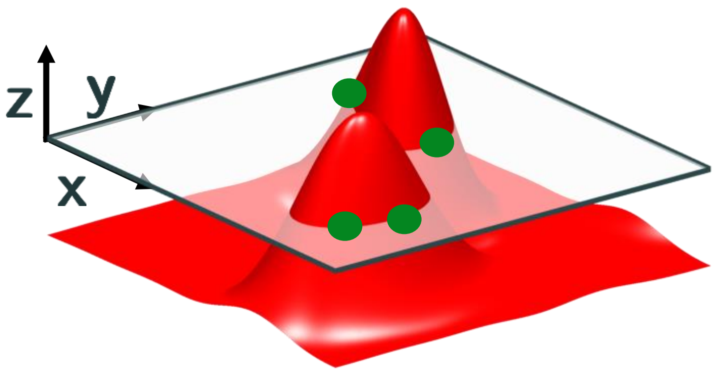
Model: Zero-level set of bandlimited function



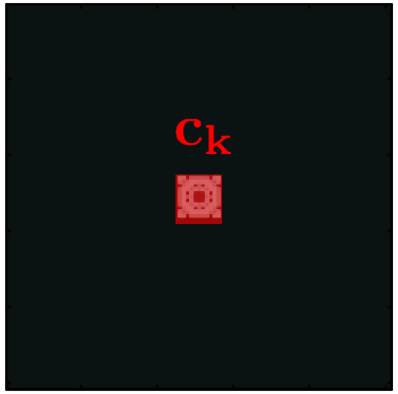
$$\{\mathbf{r} \in \mathbf{R}^n | \psi(\mathbf{r}) = 0\}$$

Zero-level set



$$\psi(\mathbf{r}) = \sum_{\mathbf{k} \in \Lambda} \mathbf{c_k} e^{j\ 2\pi \mathbf{k}^T \mathbf{r}}$$

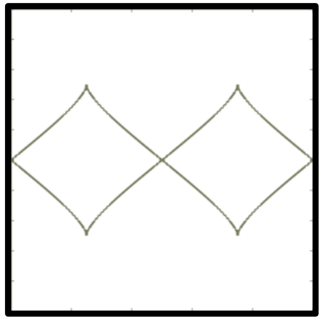
Band-limited function

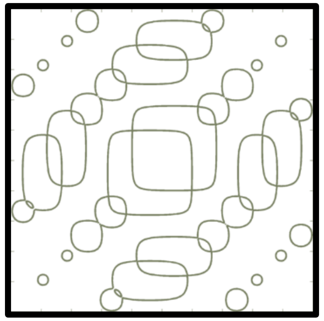


Fourier coefficients

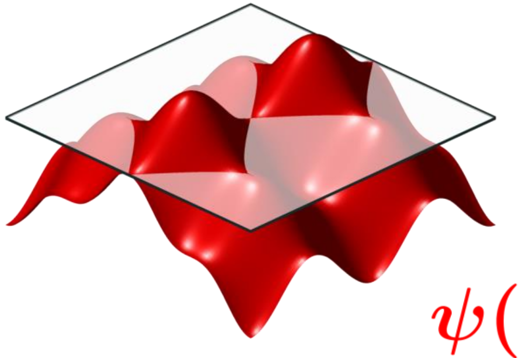
Rich enough to capture complex surfaces

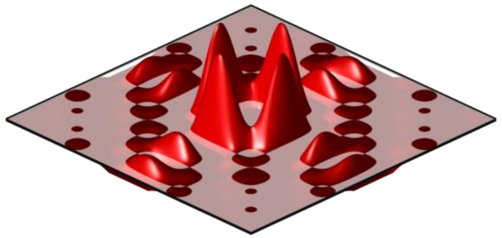
Curve complexity increases with increase in bandwidth





$$\psi(x, y) = 0$$



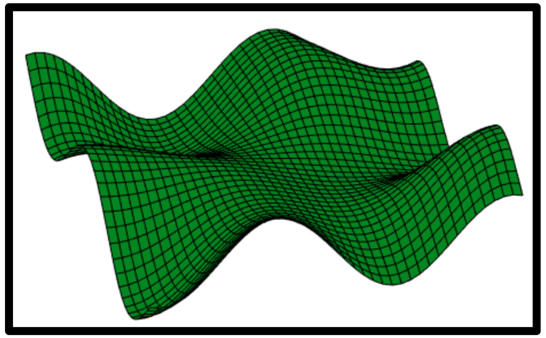


$$\psi(x, y)$$

7x9 coefficients

13x13 coefficients

Moving to higher dimensions



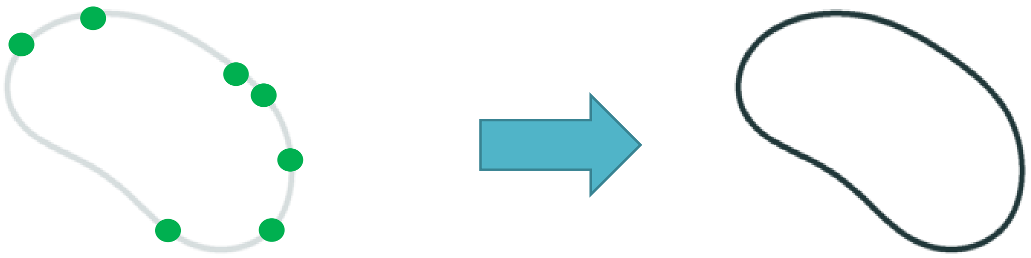
$$\psi(x, y, z) = 0$$

$$\psi(x, y, z)$$

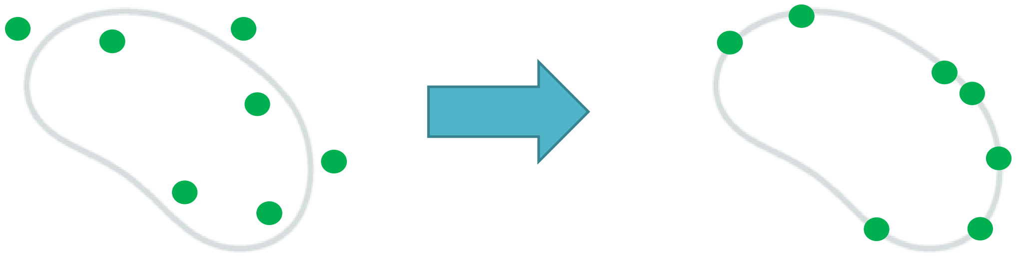
5x5x5 coefficients

Aim of this work

- Problem 1: Recovery of curves from sampled points

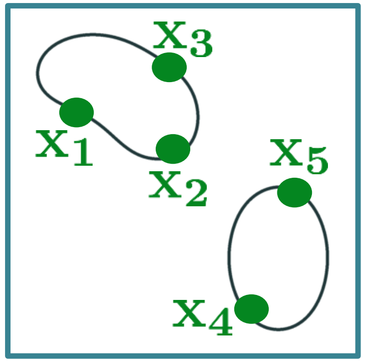


- Problem 2: Recovery of points on the curve from corrupted measurements



Model property: filter coefficients annihilate feature matrix

For N points $\{\mathbf{x}_i\}, i = 1, \ldots, N$ on the curve: $\psi(\mathbf{x}_i) = 0$



$\{\mathbf{r} \in \mathbf{R}^n | \psi(\mathbf{r}) = 0\}$

$$\sum_{\mathbf{k} \in \Lambda} \mathbf{c_k} e^{j\, 2\pi \mathbf{k}^T \mathbf{x}_i} = 0$$

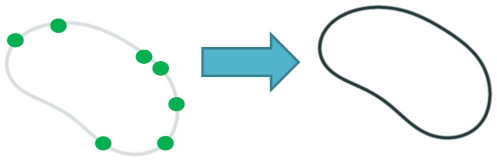
$$\mathbf{c}^T \underbrace{\begin{bmatrix} e^{j2\pi \mathbf{k}_1^T \mathbf{x}_i} \\ e^{j2\pi \mathbf{k}_2^T \mathbf{x}_i} \\ \ldots \\ e^{j2\pi \mathbf{k}_{|\Lambda|}^T \mathbf{x}_i} \end{bmatrix}}_{\phi(\mathbf{x}_i)} = 0 \quad \Rightarrow \quad \mathbf{c}^T \underbrace{\left[\phi(\mathbf{x}_1)\ \phi(\mathbf{x}_2)\ \ldots\ \phi(\mathbf{x}_N)\right]}_{\Phi(\mathbf{X})} = 0$$

Feature vector

Feature matrix

Filter $\mathbf{c}$ annihilates Feature matrix

Outline

- Union of curves model
- Recovery of curves from samples 
- Solving inverse problems using model
- Summary

Recovery of curve from samples

Model property: Filter $\mathbf{c}$ annihilates Feature matrix $\mathbf{c}^T\mathbf{\Phi}(\mathbf{X}) = \mathbf{0}$

Aim: Recover curve $\psi(\mathbf{r}) = \mathbf{0}$ from samples $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x}_N$

Step-1 Form feature matrix $\mathbf{\Phi}(\mathbf{X})$

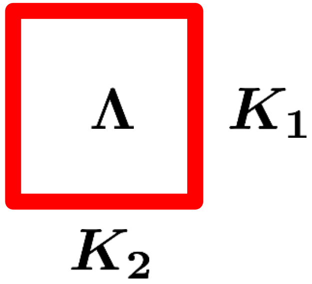
Step-2 Find null-space vector $\mathbf{c}$

Step-3 Inverse Fourier transform gives polynomial $\psi(\mathbf{r})$

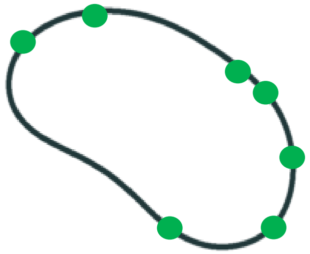
Step-4 Take zero level-set to get $\psi(\mathbf{r}) = \mathbf{0}$

Proposition-1: Number of samples for perfect recovery

Polynomial $\psi(\mathbf{r})$ with Fourier support $\boldsymbol{\Lambda}$



N points lying on
$\psi(\mathbf{r}) = 0$



How many points required to recover curve uniquely?

Case-1

$\psi(\mathbf{r})$ is irreducible

$$N \geq (K_1 + K_2)^2$$

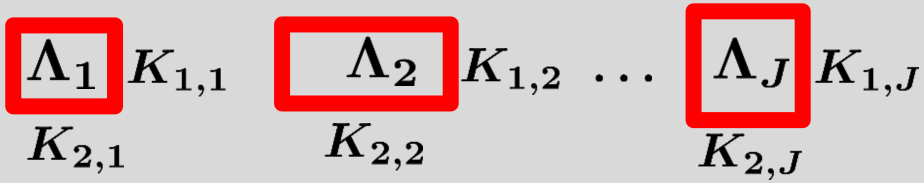
samples

Proof uses Beźout's inequality

Case-2

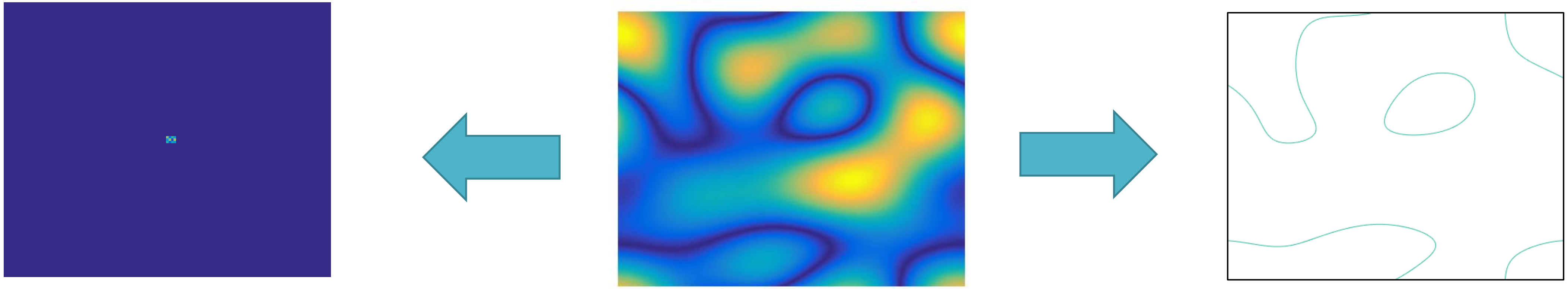
J irreducible factors

$$\psi(\mathbf{r}) = \psi_1(\mathbf{r})\psi_2(\mathbf{r})\ldots\psi_J(\mathbf{r})$$



$N_j \geq (K_1 + K_2)(K_{1,j} + K_{2,j})$ samples on j^{th} factor

Proposition-1: Number of samples for perfect recovery

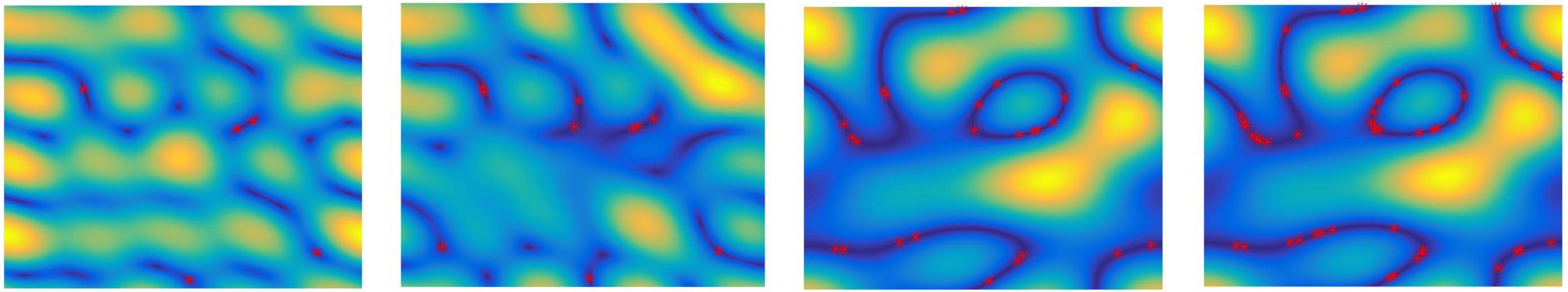


Fourier co-efficients: 5x5 support

$\psi(x,y)$

$\psi(x,y) = 0$

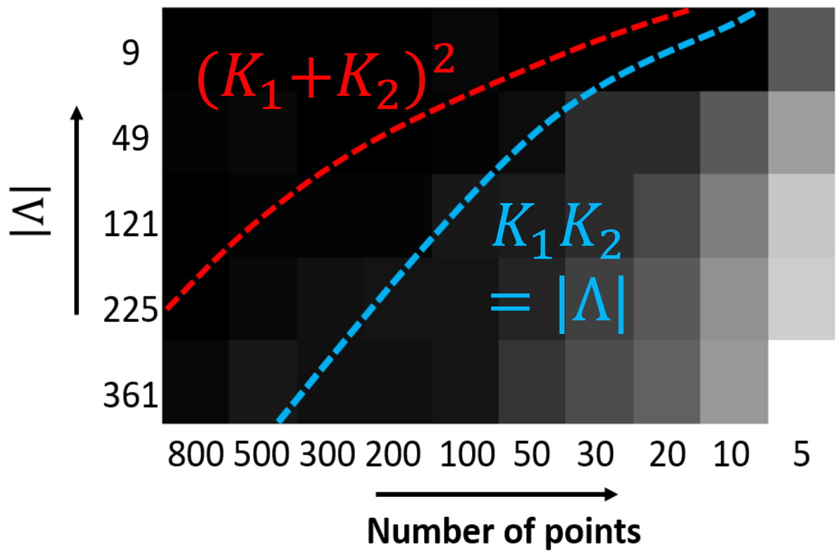
Recovery of curve from samples



5 points | 10 points | 25 points | 50 points

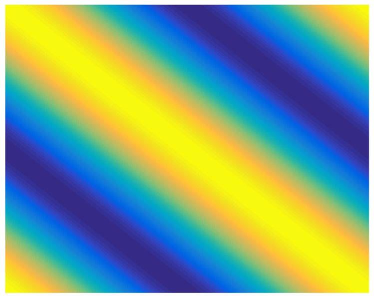
Proposition-1: Comparing to a degrees of freedom argument

Probability of failure

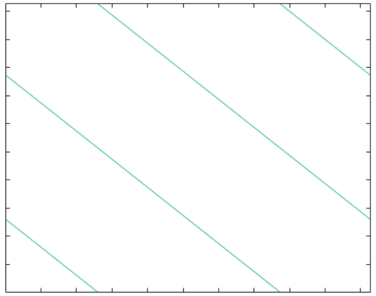


Are $|\Lambda|$ points sufficient?

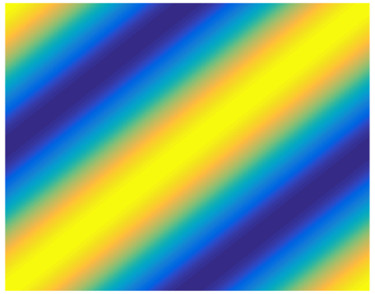
Counter-example



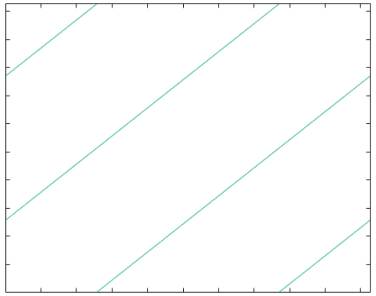
$\psi_1(x,y) \equiv cos(2\pi(x+y))$



$\psi_1(x,y) = 0$

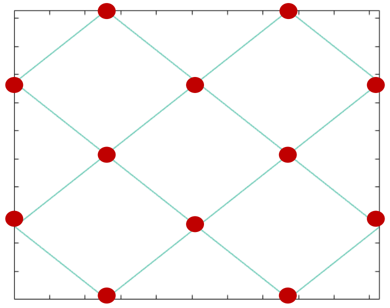


$\psi_2(x,y) \equiv cos(2\pi(x-y))$



$\psi_2(x,y) = 0$

Fourier co-efficients: 3x3 support



12 intersection points

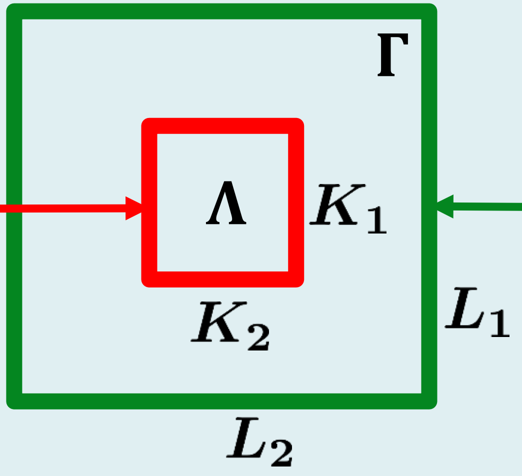
$|\Lambda| = 9$

but > 12 samples for unique recovery

Proposition-2: Recovery using over-estimated filter

Minimal filter support

Required $\boldsymbol{\Lambda}$ is unknown!



Filter support over-estimated in practice

N points lying on $\psi(\mathbf{r}) = 0$

$\psi(\mathbf{r})$ is irreducible

$$N \geq (L_1 + L_2)(K_1 + K_2)$$

Multiple solutions of form:

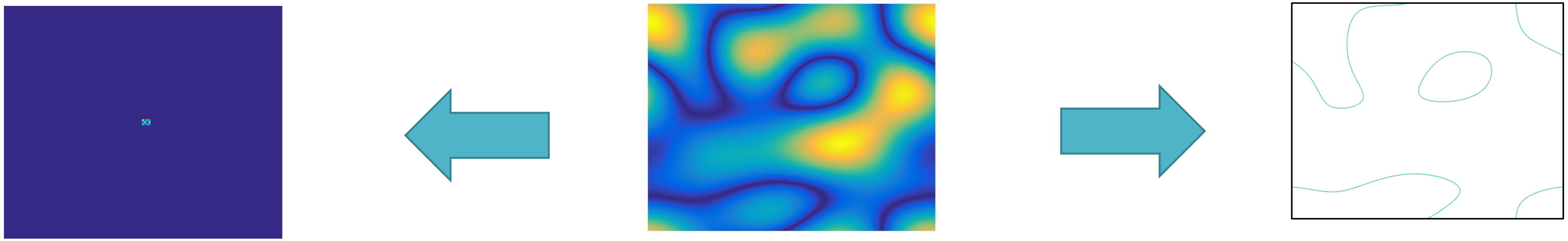
$$\hat{\psi}(\mathbf{r}) = \psi(\mathbf{r})\eta(\mathbf{r})$$

for arbitrary $\eta(\mathbf{r})$

Common zeros of all solutions give $\psi(\mathbf{r}) = 0$

Reducible polynomials: $N_j \geq (L_1 + L_2)(K_{1,j} + K_{2,j})$

Proposition-2: Recovery using over-estimated filter

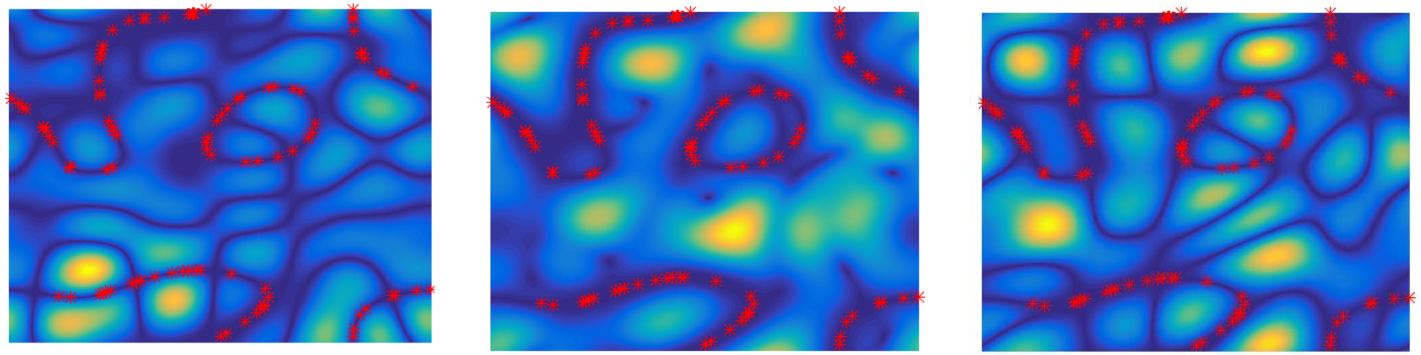


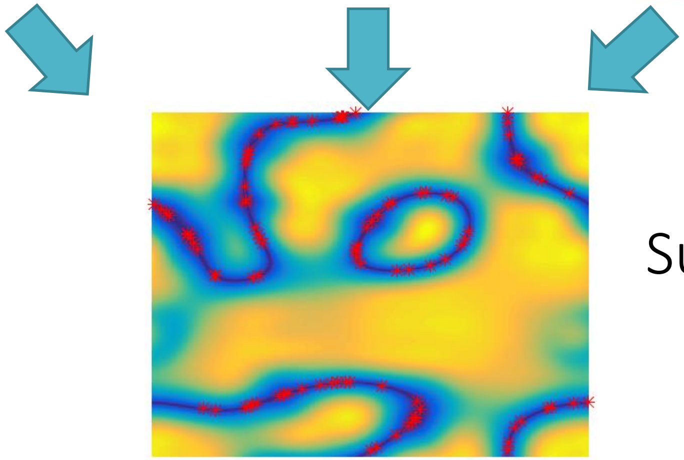
Fourier co-efficients: 5x5 support

$\psi(x,y)$

$\psi(x,y) = 0$

Recovered polynomials from 100 samples using 11x11 support





Sum-of-squares combined

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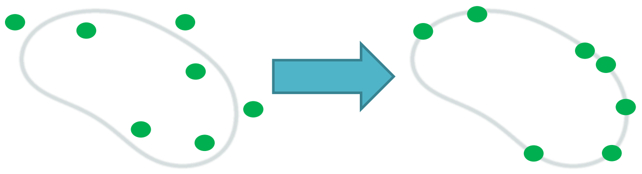
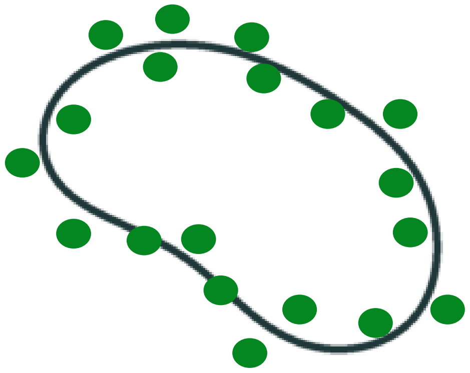
- Solving inverse problems using model 
- Summary

Solving inverse problems using model

Problem: Recover points $\{\mathbf{x}_i\}$ from corrupted measurements:

$$\mathbf{b}_i = \mathcal{A}_i(\mathbf{x}_i) + \boldsymbol{\eta}_i$$



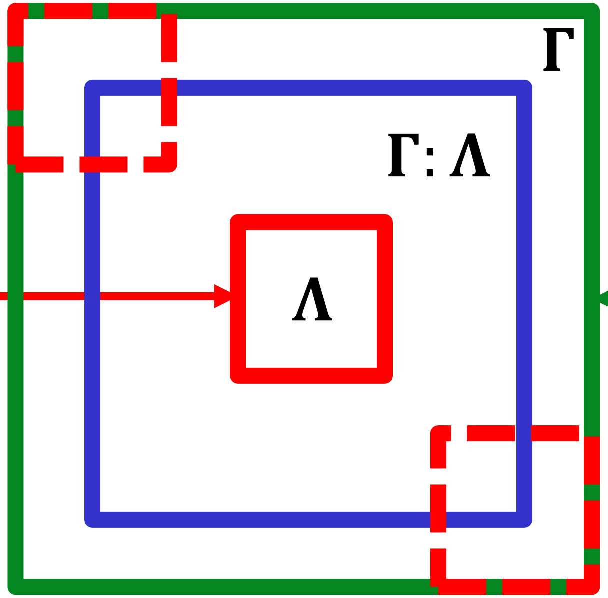
Example: Denoising problem

$\{\mathbf{b}_i\}$ lie near curve $\{\mathbf{r} \in \mathbf{R}^n | \psi(\mathbf{r}) = \mathbf{0}\}$

$\psi(\mathbf{r})$: Bandlimited function

Solution: Use model properties

Proposition-3: Rank of the feature matrix



Minimal filter support

With sufficient points $\boldsymbol{N}$:
Unique null-space vector $\mathbf{c}$

$$\text{rank}\left(\boldsymbol{\Phi}(\mathbf{X})\right) = |\boldsymbol{\Lambda}| - \mathbf{1}$$

Over-estimated filter support

With sufficient points $\boldsymbol{N}$:

$$\text{rank}\left(\boldsymbol{\Phi}(\mathbf{X})\right) = |\boldsymbol{\Gamma}| - |\boldsymbol{\Gamma} : \boldsymbol{\Lambda}|$$

Number of valid shifts of $\boldsymbol{\Lambda}$ in $\boldsymbol{\Gamma}$

Feature matrix is rank-deficient

Enforcing low rank feature matrix to solve inverse problems

$$\min_{\mathbf{X}} \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|^2 + \lambda \|\Phi(\mathbf{X})\|_*$$

Iterative reweighted least squares scheme

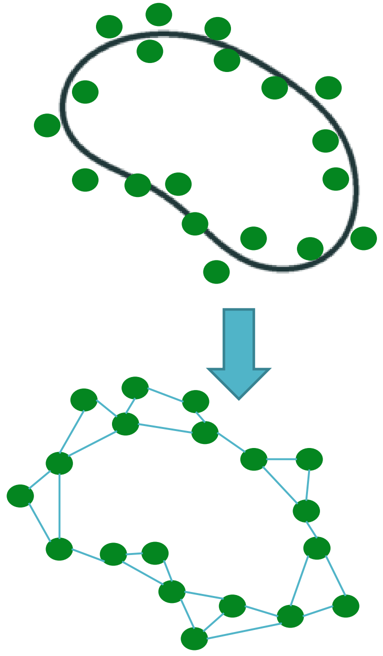
$$\mathbf{X}^{(n)} = \arg\min_{\mathbf{X}} \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|^2 + \lambda \, \mathbf{trace}[\mathcal{K}(\mathbf{X})\mathbf{Q}^{(n-1)}]$$

$$\mathbf{Q}^{(n)} = [\mathcal{K}(\mathbf{X}^{(n)}) + \gamma^{(n)}\mathbf{I}]^{-\frac{1}{2}}$$

Gradient linearization

$$\mathbf{X}^{(n)} = \arg\min_{\mathbf{X}} \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|^2 + \lambda \, \mathbf{trace}(\mathbf{X}^T\mathbf{L}^{(n-1)}\mathbf{X})$$

where $\mathbf{L}^{(n-1)} = f(\mathcal{K}(\mathbf{X}^{(n-1)}), \mathbf{Q}^{(n-1)})$

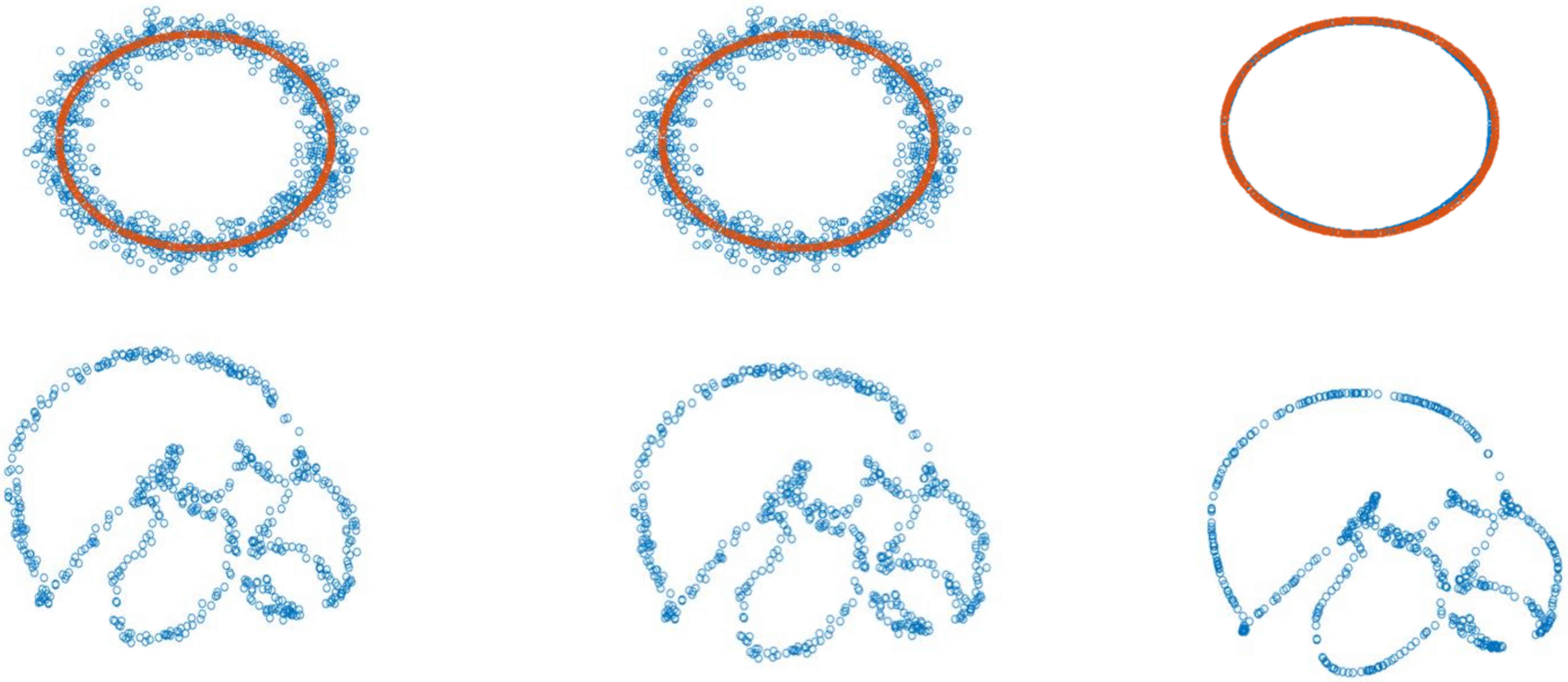


Introduced for Polynomial Kernels in “Algebraic Variety Models for High-Rank Matrix Completion”, G. Ongie et al MLR 2017

Denoising synthetic data

Circle

Tigerhawk logo



Noisy points

1^{st} iteration

50^{th} iteration

Relation between model and kernel low-rank methods

Feature matrix

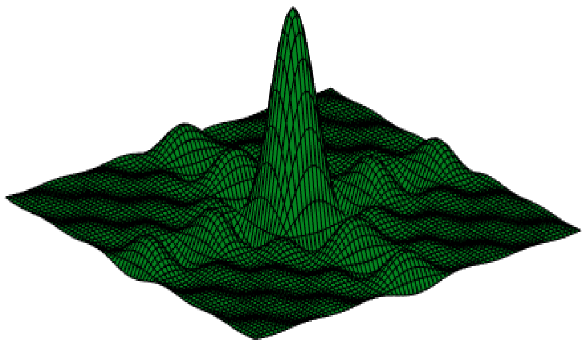
$$\mathbf{\Phi}(\mathbf{X}) = \left[\phi(\mathbf{x_1})\ \phi(\mathbf{x_2})\ \ldots\ \phi(\mathbf{x}_N)\right]$$

Size: $|\mathbf{\Gamma}| \times N$ $|\mathbf{\Gamma}|$ increases exponentially with number of dimensions

Computing the Gram matrix

$$\mathcal{K}(\mathbf{X}) = \mathbf{\Phi}(\mathbf{X})^H \mathbf{\Phi}(\mathbf{X})$$ ➡ Kernel matrix

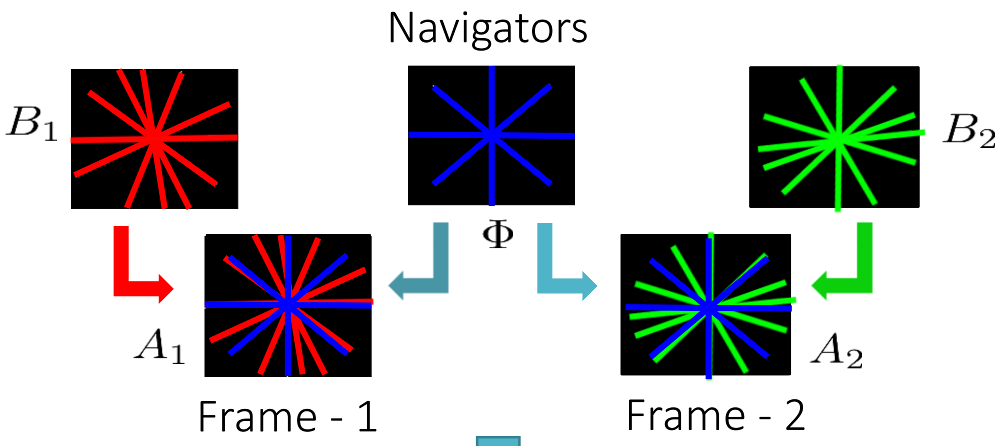
Size: $N \times N$ Size independent of ambient dimension



Dirichlet kernel

Free-breathing cardiac MR reconstruction

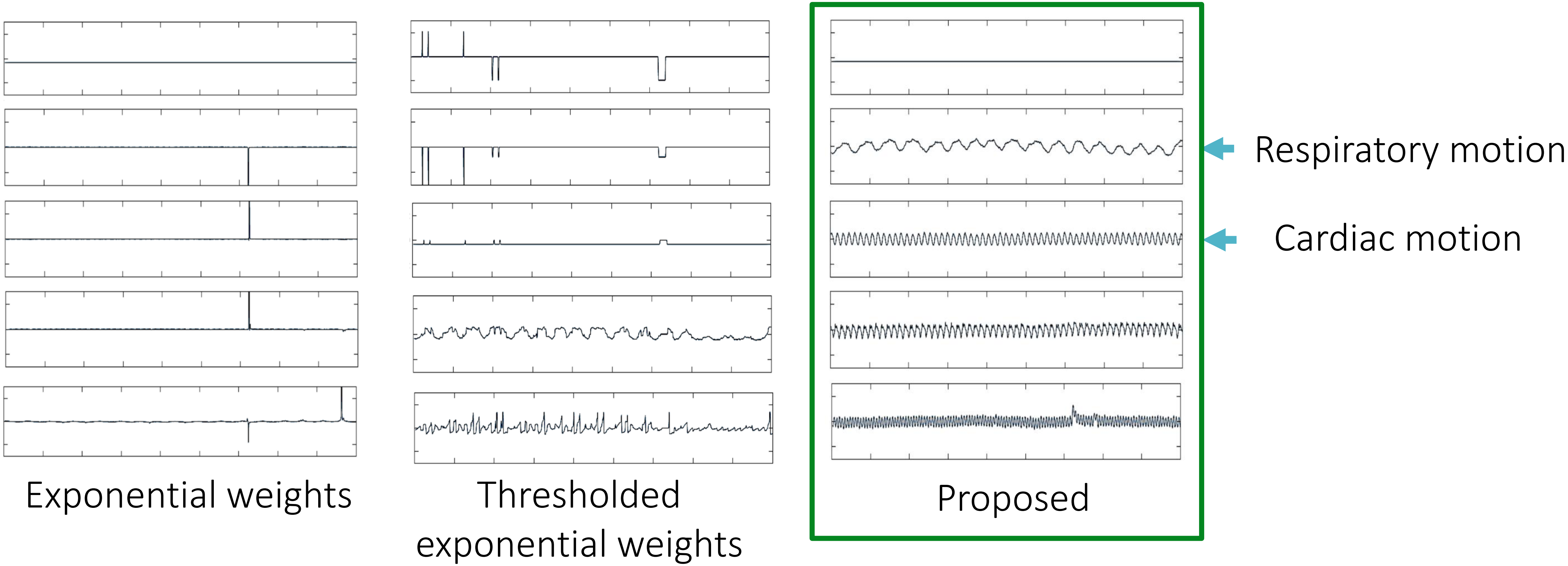
"Dynamic MRI using SToRM"
S. Poddar et al, TMI 2016



Estimate Laplacian from navigators using proposed scheme

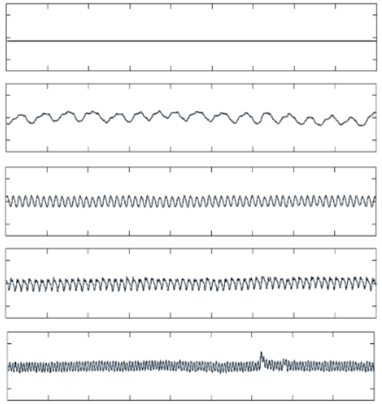
$$\min_X \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|^2 + \lambda \mathrm{Tr}(\mathbf{X}^T \mathbf{L} \mathbf{X})$$

Improved estimation of Laplacian eigen vectors



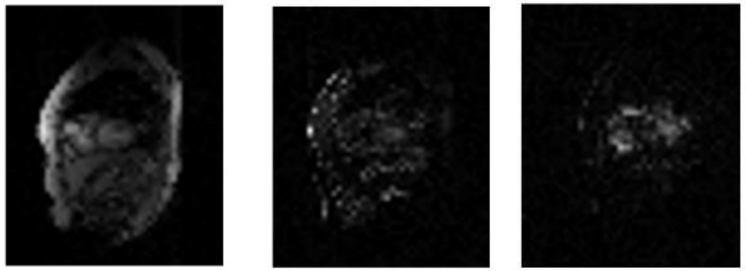
- Depends on threshold
- Does not capture physiological signal

Approximation of image series using few basis functions

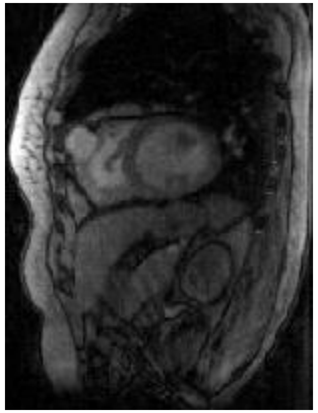


r = 30 basis functions

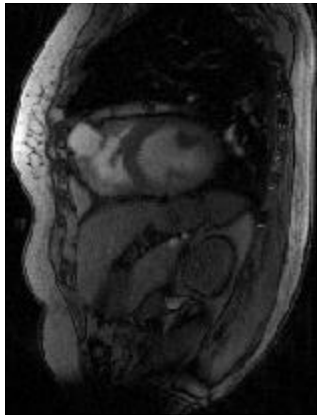
$$\mathbf{X} = \mathbf{U}_r \mathbf{V}_r$$



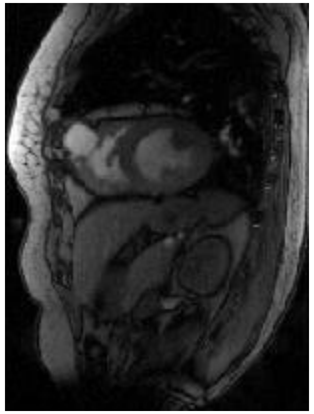
Only r basis images to be reconstructed



Exponential weights
20 min
Slow

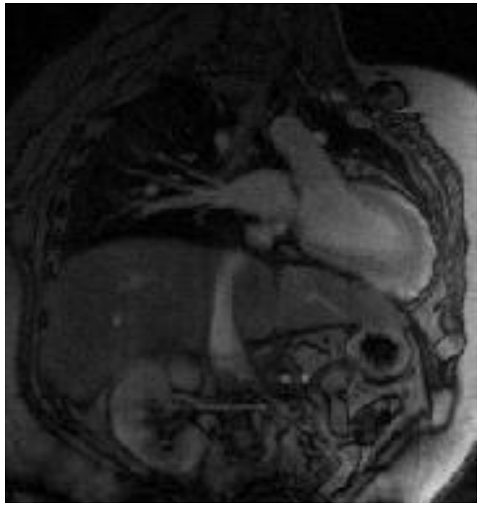


Exp Weights + UV factorization
2 min
Motion artefact

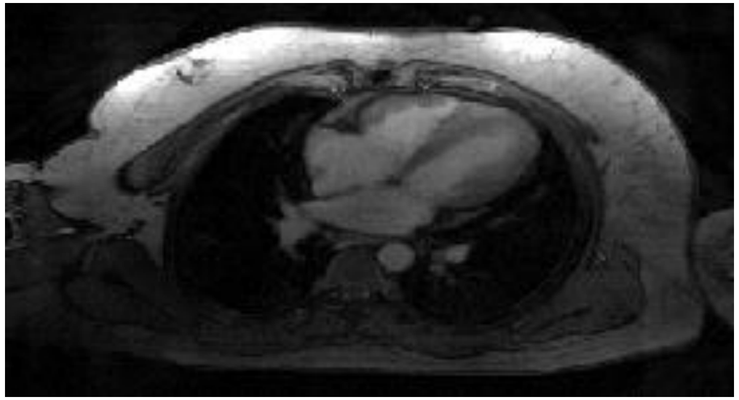


Proposed
2 min

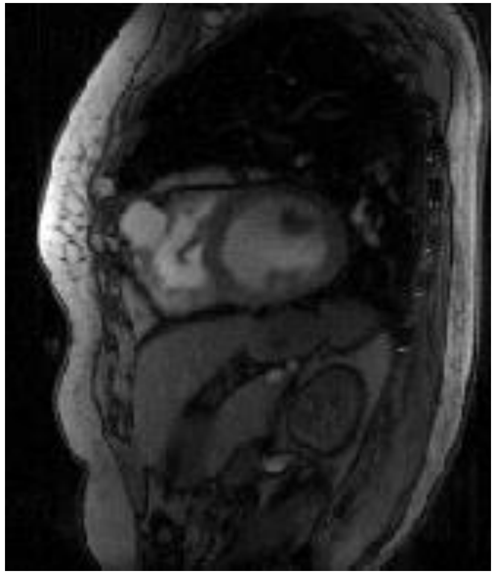
Reconstructed free-breathing cardiac datasets



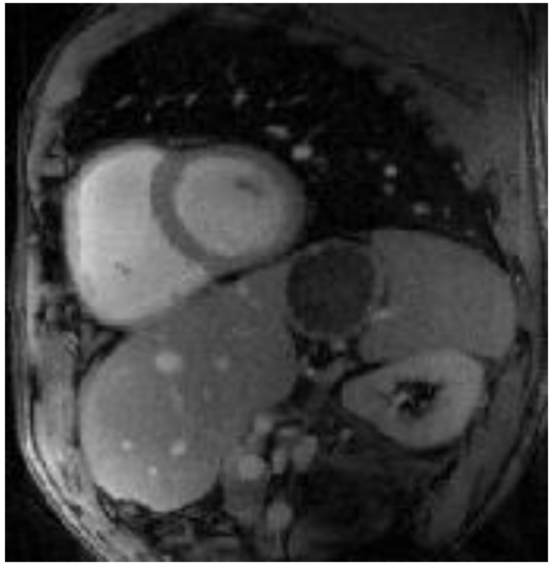
2-chamber view



4-chamber view



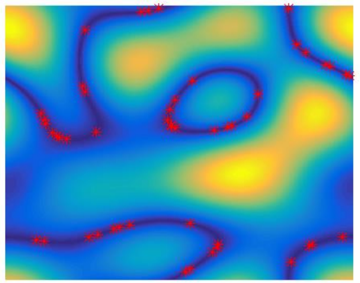
Short axis view



Short axis view

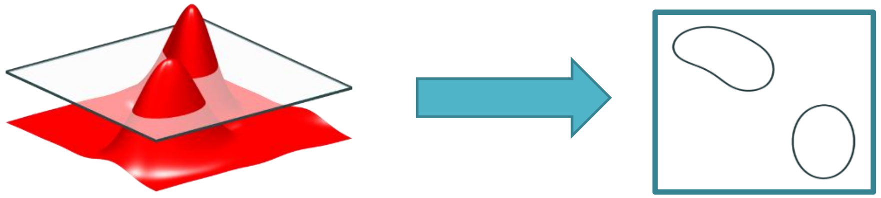
10 patients recruited at the University of Iowa Hospitals and Clinics

Summary

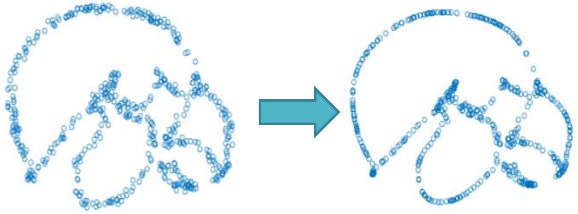


- Union of curves model

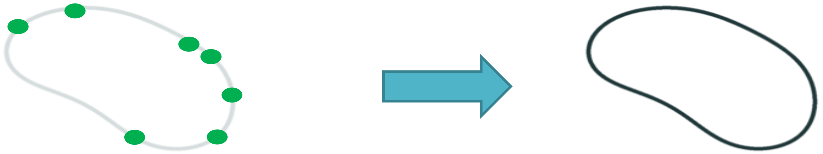
$\psi(\mathbf{r})$

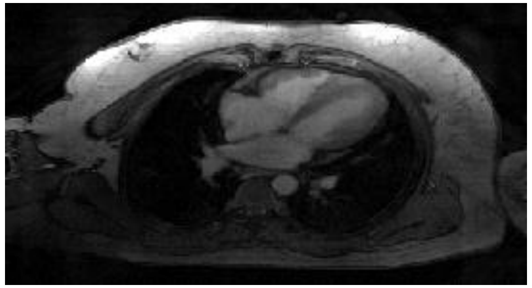


$\{\mathbf{r} \in \mathbf{R}^n | \psi(\mathbf{r}) = 0\}$

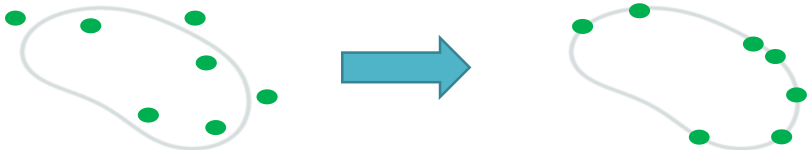


- Guarantees for **recovery of curves** from their samples

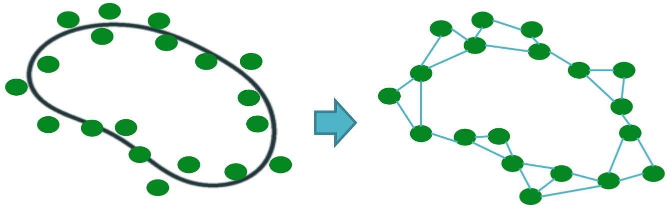




- Solving **inverse problems** using low-rank feature matrix



- Connection to **kernels** and **graph Laplacian**



Questions?