Consensus Based Carrier Synchronization in a Two Node Network [⋆]

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Abstract: Motivated by synchronous communications, we propose a consensus based carrier synchronization algorithm involving two transceiving units. Our algorithm achieves frequency lock globally and exponentially. Further it also achieves phase synchronization in the following sense. Asymptotically it induces the two transmitters to be either in phase, or out of phase by 180 degrees. We provide a simple method for the transmitters to know if they are out phase by 180 degrees. This constitutes a significant advance over existing carrier synchronization technology which is largely based on Phase Locked Loops that only achieve lock locally.

Keywords: Consensus, Cooperative control, Global Stability, Synchronization, Phase Locked Loop

1. INTRODUCTION

This paper enunciates a globally stable nonlinear consensus based algorithm that achieves carrier synchronization between two cooperating transmitters. Here carrier synchronization refers to frequency and phase synchronization.

The setting we consider involves two transmitters. Each transmits to the other its sinusoidal carrier, and adjusts its frequency and phase to achieve global consensus. In particular depending on a design parameter α, both carriers globally attain a common frequency that is an integer multiple of π/α. As explained in the sequel, even with $\alpha = 1$ second, such a granularity is entire acceptable. As also explained in the sequel a larger α does not cause performance problems, and thus arbitrary granularity can be achieved at no practical expense.

The steady state phase offset induced by the algorithm is an integer multiple of π. Should the multiple be even, then this corresponds to a complete phase lock. In many applications a phase disparity of π is completely acceptable. Where exact phase lock is needed, it turns out that the transmitters can determine whether a phase disparity of π exists simply from the consensus frequency they attain, and can thus correct for the phase discrepancy.

This algorithm thus represents a very substantial improvement on existing carrier synchronization methods that largely employ Phase Locked Loop (PLL) technology, Lathi and Ding [2008]. As is well known PLL's achieve only local carrier synchronization. Despite having sinusoids in its update kernel our algorithm is globally stable.

Carrier synchronization is critical to virtually all synchronous communications tasks, Lathi and Ding [2008].

In a standard communication system the receiver and transmitter must have carrier lock to achieve high fidelity demodulation. This is also an instance where a phase disparity of π is as good as a phase disparity of zero.

Carrier synchronization is also essential for beamforming, Mudumbai et. al. [2006]- Mudumbai et. al. [2010], that is an important part of the emerging field of cooperative communications, Sendonaris et al. [2003], Erkip et al. [2004], Nostratinia, et. al. [2004], Jayaweera [2006], Scaglione et. al. [2007]. In this case two nodes must achieve phase and frequency synchronization for cooperative transmission that causes their respective carrier powers to reinforce each other. Beamforming algorithms, e.g. Mudumbai et. al. [2006]- Mudumbai et. al. [2010], assume *a priori* frequency synchronization which our algorithm provides.

At another level this paper should be viewed as a contribution to the multiagent consensus literature, Yamaguchi and Beni [1996], Toner and Tu [1998], Vicsek et al. [1995], Jadbabie et al. [June 2003], Lin et. al. [2004] and Ren and Beard [2007], or the related synchronization literature. The latter includes the synchronized flashing of fireflies Mirollo and Strogatz [1990] that has motivated other researchers to propose similar methods to achieve synchronization in wireless networks, Werner-Allen et. al. [2005]. Other examples include the study of various biological phenomena, Pittendrigh [1960], Buck [1988] and Murthy and Fetz [1996], network time synchronization, Cavendish [2000], Butterline and Frodge [1999], and Nelson et. al. [2002] and on-chip clock distribution, Hemani et. al. [2000].

Section 2 describes our synchronization algorithm and its implementation. Another candidate for frequency synchronization is the Kuramoto algorithm Acebrón et. al. [2005]. In section 3 we compare our algorithm with Kuramoto. Section 4 characterizes all locally stable stationary trajec-

tories. Section 5 provides global stability analysis. Section 6 concludes.

2. THE ALGORITHM

Consider two transmitters that must achieve frequency lock. To this end they continuously broadcast to each other their current carrier and must adjust their carriers to achieve a consensus frequency.

In particular assume that the i-th agent broadcasts a signal $\cos(\theta_i(t))$. Such an agent receives a signal $j \neq i$,

$$s_i(t) = A\cos(\theta_j(t)) + v_i(t), \tag{1}$$

where A reflects the attenuation suffered in the transmission from agent j to i and $v_i(t)$ is noise. In an uncluttered environment it is reasonable to assume that the attenuation suffered by both is the same.

The algorithm we propose is as follows. For $\beta, \alpha > 0$, $\{i, j\} = \{1, 2\}$ and $j \neq i$

$$\dot{\theta}_i(t) = \omega_i(t) \tag{2}$$

$$\dot{\omega}_i(t) = -\beta A \sin\left(\theta_i(t) - \theta_j(t) + \alpha\omega_i(t)\right). \tag{3}$$

Thus $\omega_i(t)$ represents the locally generated instantaneous frequency. The $\alpha\omega_i(t)$ term in the frequency update equation in (3) is extremely crucial to our synchronization algorithm, because it ensures the stability of the consensus solution. An obvious discrete time counterpart of (3) is as follows. For small time step Δ:

$$\theta_i(t + \Delta) = \theta_i(t) + \Delta\omega_i(t), \tag{4}$$

$$\omega_i(t + \Delta) = \omega_i(t) - \beta\Delta A \sin\left(\theta_i(t) - \theta_j(t) + \alpha\omega_i(t)\right) \tag{5}$$

It is clear that the qualitative properties of (4, 5) approach those of (2, 3) for small time steps Δ. For larger Δ, the same effect can be achieved by scaling down β.

We next turn to the implementation of the algorithm given that the information available to agent i, is the signal generated by the other agent, exemplified by (1), its instantaneous frequency $\omega_i(t)$ and the instantaneous phase $\theta_i(t)$. We first note that in practice, though not in concept, all signals will be translated from RF to IF or baseband. Assume that the difference between initial instantaneous frequencies are small compared to their values. In other words both $\sin\left(\theta_i(t) - \theta_j(t)\right)$ and $\cos\left(\theta_i(t) - \theta_j(t)\right)$ represent relatively low pass signals as compared to $\sin\left(\theta_i(t) + \theta_j(t)\right)$ and $\cos\left(\theta_i(t) + \theta_j(t)\right)$.

Then consider the setting of Figure 1 where the blocks labeled LPF are low pass filters. Observe as $\theta_i(t)$ and $\omega_i(t)$ are available to agent i, one can generate:

$$\begin{aligned} g_i(t) &= 2s_i(t)\cos(\theta_i(t)) \\ &= A\left[\cos(\theta_i(t) - \theta_j(t)) + \cos(\theta_i(t) + \theta_j(t))\right] \\ &\quad + 2v_i(t)\cos(\theta_i(t)). \end{aligned} \tag{6}$$

Thus, to within a noise perturbation, the low pass filtered version of g_i and by similar anlaysis, that of $2s_i(t)\sin(\theta_i(t))$ are respectively given by

$$A\cos(\theta_i(t) - \theta_j(t)) \text{ and}$$
$$A\sin(\theta_i(t) - \theta_j(t)).$$

Thus, as $\omega_i(t)$ is available, one can indeed as per Figure 1 generate the kernel in (3) to within a perturbing noise. *It is also clear that should the noise $v_i(t)$, be white Gaussian, so is the noise perturbing the kernel of (3).* Further the net noise is additive and is the original noise scaled by 2β.

Note this implementation assumes that the initial frequency disparity is small relative to the actual frequencies. To be more concrete let us consider some realistic numbers. In practice the nominal frequencies would be at RF, i.e. hundreds of MHz or even GHz. Frequency disparities on the other hand would be in *at most* hundreds of Hz. Thus the components to be retained at the output of the LPF's in fig. 1 have frequencies (hundreds of Hz) that are orders of magnitude lower than the frequencies (at least hundreds of MHz) of the components to be filtered out. Thus the filtering can be very effectively implemented.

This also emphasizes the need for stability that is not merely local, as initial frequency errors could be nontrivial, as high as hundreds of Hz. In practice, (2, 3) or indeed (4,5) will be implemented at baseband, i.e. tens of KHz. This will involve a standard frequency translation prior to implementation.

3. COMPARISON WITH KURAMOTO

One well studied algorithm that can achieve frequency synchronization is the Kuramoto algorithm, Acebrón et. al. [2005]. Translated to a two node network it becomes for $\{i, j\} = \{1, 2\}$, $i \neq j$,

$$\dot{\theta}_i = \omega_i + K\sin(\theta_j - \theta_i), \tag{7}$$

where K is a coupling parameter, θ_i and ω_i are the instantaneous phase and the initial frequency estimate of node i's oscillator signal, respectively. Frequency synchronization is achieved if for all i, j

$$\dot{\theta}_i = \dot{\theta}_j. \tag{8}$$

We now reveal a key difficulty with (7) to carrier frequency synchronization. For the $\dot{\theta}_i$ to synchronize we need

$$\omega_1 + K\sin(\theta_2 - \theta_1) = \omega_2 + K\sin(\theta_1 - \theta_2).$$

At the minimum this requires that

$$2K \geq |\omega_1 - \omega_2|. \tag{9}$$

In fact *the actual bound needed for stable synchronization is significantly higher, Jadbabaie et al. [June 2004], Dorfler and Bullo [2010].* Thus the coupling coefficient K must be large. The implementation of (7) would involve similar procedures as those for (2,3). As a consequence it can be verified that in this implementation the noise gets amplified by $2K$. Thus unless the initial frequencies are sufficiently close, not only will Kuramoto stabilize only at the expense of noise amplification, but will even lack a well defined consensus state, unless K is significantly large.

By contrast as shown in the next section, the equilibrium trajectories of (2, 3) are independent of β.

Now consider Figure 2. This depicts the frequency plot generated by our algorithm in a 2-node network with initial frequencies at 2000 and 2100 radians/sec, initial phase difference of $\pi/4$ radians, with $\beta = 1$, $\alpha = 1$ and demonstrates synchronization. Thus the noise amplification factor in this case is 2. On the other hand (7) would
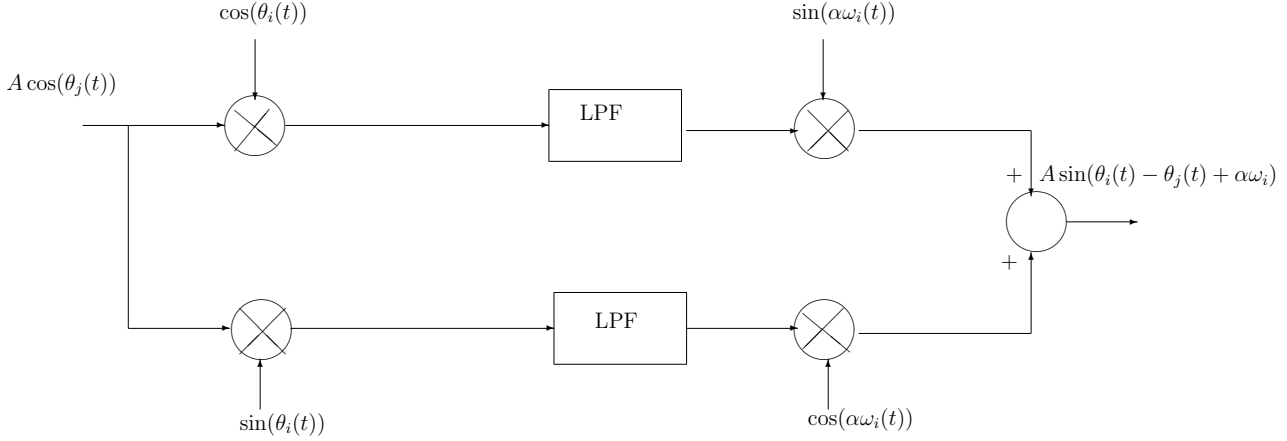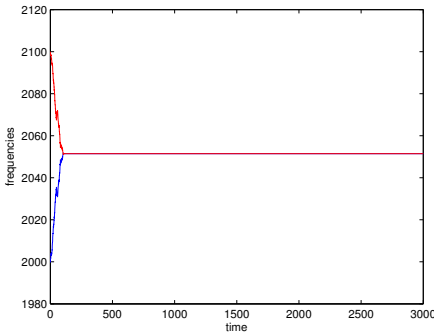
Fig. 1. Implementation

Fig. 2. Convergence with $\omega_1(0) = 2000$ rad/sec and $\omega_2(0) = 2100$, and $\beta = 1$ and $\alpha = 1$.

require a $K = 50$. Thus our algorithm synchronizes with a much smaller noise amplification factor than required by Kuramoto to even have a well defined consensus state, let alone stablity.

4. LOCALLY STABLE CONSENSUS STATES

The two node algorithm is with $\alpha, \beta > 0$ to: for $i \in \{1, 2\}$ (2) and:

$$\dot{\omega}_1(t) = -\beta \sin \left(\theta_1(t) - \theta_2(t) + \alpha \omega_1(t) \right) \quad (10)$$

and

$$\dot{\omega}_2(t) = -\beta \sin \left(\theta_2(t) - \theta_1(t) + \alpha \omega_2(t) \right) \quad (11)$$

hold.

We say that consensus is achieved if there hold:

$$\sin \left(\theta_1(t) - \theta_2(t) + \alpha \omega_1(t) \right) = 0, \quad \forall t \quad (12)$$

and

$$\sin \left(\theta_2(t) - \theta_1(t) + \alpha \omega_2(t) \right) = 0, \quad \forall t, \quad (13)$$

We begin by characterizing the consensus states for the algorithm.

Theorem 4.1. Consider (2) for $i \in \{1, 2\}$, and (10) and (11) with $\beta, \alpha > 0$. The only equilibrium trajectories for this system are: for integers m and n, and any ϕ,

$$\theta_1(t) = \frac{(m+n)\pi}{2\alpha} t + \phi + \frac{(m-n)\pi}{2}, \quad (14)$$

$$\theta_2(t) = \frac{(m+n)\pi}{2\alpha} t + \phi \quad (15)$$

$$\omega_1(t) = \omega_2(t) = \frac{(m+n)\pi}{2\alpha}. \quad (16)$$

Proof: Equilibrium requires that (12) and (13) hold and because of (10) and (11), for some ω_i^*, $i \in \{1, 2\}$

$$\omega_i(t) = \omega_i^* \quad \forall t. \quad (17)$$

Thus, as $\theta_i(t)$ and $\omega_i(t)$ are both continuous, on this trajectory for some integer l, for all t, there holds:

$$\theta_2(t) - \theta_1(t) + \alpha \omega_2^* = \theta_1(t) - \theta_2(t) + \alpha \omega_1^* + l\pi$$

$$\Leftrightarrow \theta_2(t) - \theta_1(t) = \frac{\alpha \left(\omega_1^* - \omega_2^* \right) + l\pi}{2}. \quad (18)$$

Further because of (17) and (2) for all t and $i \in \{1, 2\}$, on this trajectory for constant ϕ_i:

$$\theta_i(t) = \omega_i^* t + \phi_i. \quad (19)$$

Because of (18) this must mean that for some ω^*

$$\omega_1^* = \omega_2^* = \omega^*.$$

Finally for some integers m, n

$$\phi_2 - \phi_1 + \alpha \omega^* = n\pi$$

and

$$\phi_1 - \phi_2 + \alpha \omega^* = m\pi.$$

Thus (14) to (16) hold.

∎

We next show through a linearized analysis that certain consensus frequency and phase combinations are locally unstable and some others are stable.

Theorem 4.2. Consider (2) for $i \in \{1, 2\}$, (10) and (11) with $\beta, \alpha > 0$. Then the equilibrium trajectories characterized in Theorem 4.1 are locally exponentially stable iff m and n are both even. If either m and/or n is odd then the trajectory is unstable.

Proof:

For integer m, n, define

$$\tilde{\theta}(t) = \theta_1(t) - \theta_2(t) - \frac{(m-n)\pi}{2} \quad (20)$$

for $i \in \{1, 2\}$,

$$\tilde{\omega}_i(t) = \omega_i(t) - \frac{(m+n)\pi}{2\alpha} \quad (21)$$

and

$$\tilde{\omega}(t) = \omega_1(t) - \omega_2(t). \tag{22}$$

Evidently such a trajectory is a stationary trajectory if for all t

$$[\tilde{\theta}(t), \tilde{\omega}_1(t), \tilde{\omega}_2(t)]' = 0.$$

Further by subtracting (2) for $i = 2$ from (2) for $i = 1$, one obtains:

$$\dot{\tilde{\theta}}(t) = \tilde{\omega}(t), \tag{23}$$

Also from (10) one obtains:

$$
\begin{aligned}
\dot{\tilde{\omega}}_1(t) &= -\beta \sin\left(\theta_1(t) - \theta_2(t) + \alpha\omega_1(t)\right) \\
&= -\beta \sin\left(\theta_1(t) - \theta_2(t) - \frac{m-n}{2}\pi + \frac{m-n}{2}\pi \right. \\
&\quad \left. + \alpha\omega_1(t) - \frac{m+n}{2}\pi + \frac{m+n}{2}\pi\right) \\
&= -\beta \sin\left(\tilde{\theta}(t) + m\pi + \alpha\tilde{\omega}_1(t)\right) \\
&= -(-1)^m \beta \sin\left(\tilde{\theta}(t) + \alpha\tilde{\omega}_1(t)\right)
\end{aligned} \tag{24}
$$

Similarly, from (11) one obtains:

$$
\begin{aligned}
\dot{\tilde{\omega}}_2(t) &= -\beta \sin\left(\theta_2(t) - \theta_1(t) + \alpha\omega_2(t)\right) \\
&= -\beta \sin\left(\theta_2(t) - \theta_1(t) + \frac{m-n}{2}\pi - \frac{m-n}{2}\pi \right. \\
&\quad \left. + \alpha\omega_2(t) - \frac{m+n}{2}\pi + \frac{m+n}{2}\pi\right) \\
&= -\beta \sin\left(-\tilde{\theta}(t) + n\pi + \alpha\tilde{\omega}_2(t)\right) \\
&= -(-1)^n \beta \sin\left(-\tilde{\theta}(t) + \alpha\tilde{\omega}_2(t)\right)
\end{aligned} \tag{25}
$$

Linearizing (23), (24) and (25) around zero, we obtain (23),

$$\dot{\tilde{\omega}}_1(t) = -(-1)^m \beta \left(\tilde{\theta}(t) + \alpha\tilde{\omega}_1(t)\right) \tag{26}$$

and:

$$\dot{\tilde{\omega}}_2(t) = -(-1)^n \beta \left(-\tilde{\theta}(t) + \alpha\tilde{\omega}_2(t)\right). \tag{27}$$

We now consider two cases that exhaust all possibilities.

Case I: Either both m and n are even, or they are both odd. In this case subtracting (27) from (26) we get

$$
\begin{bmatrix} \dot{\tilde{\theta}}(t) \\ \dot{\tilde{\omega}}(t) \end{bmatrix} = (-1)^m \begin{bmatrix} 0 & 1 \\ -2\beta & -\beta\alpha \end{bmatrix} \begin{bmatrix} \tilde{\theta}(t) \\ \tilde{\omega}(t) \end{bmatrix}. \tag{28}
$$

Clearly (28) is exponentially stable iff m and thus n are both even, and is unstable if both are odd.

Case II: One among m and n is even and the other is odd. In this case because of the underlying symmetries we can without loss of generality assume that m is even and n is odd. Then (23), (26) and (27) become:

$$
\begin{bmatrix} \dot{\tilde{\theta}}(t) \\ \dot{\tilde{\omega}}_1(t) \\ \dot{\tilde{\omega}}_2 1 t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -\beta & -\beta\alpha & 0 \\ -\beta & 0 & \beta\alpha \end{bmatrix} \begin{bmatrix} \tilde{\theta}(t) \\ \tilde{\omega}_1(t) \\ \tilde{\omega}_2(t) \end{bmatrix}. \tag{29}
$$

Now observe that

$$
\begin{bmatrix} 0 & 1 & -1 \\ -\beta & -\beta\alpha & 0 \\ -\beta & 0 & \beta\alpha \end{bmatrix}
$$

has determinant $2\beta^2\alpha \neq 0$. Further its trace is zero. Consequently it must have an eigenvalue in the open right half plane, and thus (29) is unstable.

∎

Observe the stable frequencies are thus multiples of π/α. The potential steady state phase offsets (modulo 2π) are 0 and π. Sometimes, e.g. in a standard communications framework a phase difference that is an odd multiple of π is entirely acceptable. Nonetheless it would be useful to easily determine whether the achieved phase discrepancy is an odd multiple of π. The following lemma helps in detecting such a disparity, should it occur.

Lemma 4.1. With even integers m and n, consider

$$k = \frac{m+n}{2} \text{ and } l = \frac{m-n}{2}.$$

Then both k and l are integers and k is even iff l is even.

Proof: That with even m and n, k and l are integers is self evident. Now l is odd iff for some integer i,

$$
\begin{aligned}
& \frac{m-n}{2} = 2i + 1 \\
\Leftrightarrow{}& m - n = 2(2i + 1) \\
\Leftrightarrow{}& m = n + 2(2i + 1) \\
\Leftrightarrow{}& k = \frac{m+n}{2} = n + (2i + 1).
\end{aligned}
$$

Then the result follows as n is even. ∎

In view of this lemma and Theorem 4.2 a phase offset that is an odd multiple of π will occur iff the locally consensus frequency one achieves is an odd multiple of π/α. Should that happen, one of the nodes can simply advance its phase by π and *de facto phase as well as frequency lock is achieved.*

5. GLOBAL STABILITY

In this section we prove the global stability of (2,3). As noted in the introduction this represents a substantial advancement over existing technology. To be specific current carrier synchronization between a transmitter and a receiver, is effected using standard PLL technology. In a PLL one node transmits its carrier to a receiver, which adjusts its frequency/phase to match the transmitter's frequency. The transmitter does not adjust its carrier. Consequently, unless the phase and the frequency of the receiver are sufficiently close to that of the transmitter, frequency/phase lock will not eventuate. By contrast our algorithm requires both nodes to adjust their carriers, and forces them to achieve a consensus.

We observe that the system is autonomous. It is thus the type of system that is potentially amenable to analysis by Lasalle's Theorem, Khalil [2002]. However, Lasalle's Theorem requires a positively invariant set that is compact. There are technical difficulties with this requirement, as even after consensus is achieved, the $\theta_i(t)$ do not belong to a compact set. To circumvent this difficulty we propose an alternative, related state space for which compactness is easier to prove. Indeed this is a fifth order state space for which the state elements z_i are as below.

$$z_1(t) = \sin(\theta_1(t) - \theta_2(t) + \alpha\omega_1(t)), \qquad (30)$$

$$z_2(t) = \sin(\theta_2(t) - \theta_1(t) + \alpha\omega_2(t)), \qquad (31)$$

$$z_3(t) = \cos(\theta_1(t) - \theta_2(t) + \alpha\omega_1(t)), \qquad (32)$$

$$z_4(t) = \cos(\theta_2(t) - \theta_1(t) + \alpha\omega_2(t)), \qquad (33)$$

and

$$z_5(t) = \omega_1(t) - \omega_2(t). \qquad (34)$$

Under the two node system equations we obtain:

$$\dot{z}_1(t) = \cos(\theta_1(t) - \theta_2(t) + \alpha\omega_1(t))\,(\omega_1(t) - \omega_2(t) + \alpha\dot{\omega}_1(t))$$
$$= z_3(t)\,(z_5(t) - \beta\alpha z_1(t))\,, \qquad (35)$$

$$\dot{z}_2(t) = \cos(\theta_2(t) - \theta_1(t) + \alpha\omega_2(t))\,(\omega_2(t) - \omega_1(t) + \alpha\dot{\omega}_2(t))$$
$$= z_4(t)\,(-z_5(t) - \beta\alpha z_2(t))\,, \qquad (36)$$

$$\dot{z}_3(t) = -\sin(\theta_1(t) - \theta_2(t) + \alpha\omega_1(t))\,(\omega_1(t) - \omega_2(t) + \alpha\dot{\omega}_1(t))$$
$$= -z_1(t)\,(z_5(t) - \beta\alpha z_1(t))\,, \qquad (37)$$

$$\dot{z}_4(t) = -\sin(\theta_2(t) - \theta_1(t) + \alpha\omega_2(t))\,(\omega_2(t) - \omega_1(t) + \alpha\dot{\omega}_2(t))$$
$$= -z_2(t)\,(-z_5(t) - \beta\alpha z_2(t))\,, \qquad (38)$$

and

$$\dot{z}_5(t) = -\beta(z_1(t) - z_2(t)). \qquad (39)$$

We now analyze the stability of the system represented by (35)-(39) regardless of its origins, i.e. the tie to our algorithm.

Lemma 5.1. With $z = [z_1, \cdots, z_5]' : \mathbb{R} \rightarrow \mathbb{R}^5$, consider the system represented by (40) to (44) below.

$$\dot{z}_1(t) = z_3(t)\,(z_5(t) - \beta\alpha z_1(t))\,, \qquad (40)$$

$$\dot{z}_2(t) = z_4(t)\,(-z_5(t) - \beta\alpha z_2(t))\,, \qquad (41)$$

$$\dot{z}_3(t) = -z_1(t)\,(z_5(t) - \beta\alpha z_1(t))\,, \qquad (42)$$

$$\dot{z}_4(t) = -z_2(t)\,(-z_5(t) - \beta\alpha z_2(t))\,, \qquad (43)$$

and

$$\dot{z}_5(t) = -\beta(z_1(t) - z_2(t)). \qquad (44)$$

Then $z(t)$ is bounded and converges uniformly asymptotically to:

$$z_1 \equiv 0 \qquad (45)$$

and

$$z_2 \equiv 0. \qquad (46)$$

Proof: Since the system of equations under consideration is autonomous, asymptotic stability implies uniform asymptotic stability.

Observe first that (dropping the argument t)

$$\frac{d}{dt}\left(z_1^2 + z_3^2\right) = 2\,(z_1\dot{z}_1 + z_3\dot{z}_3)$$
$$= 2\,(z_1 z_3(t)\,(z_5(t) - \beta\alpha z_1(t))$$
$$- z_3 z_1(t)\,(z_5(t) - \beta\alpha z_1(t)))$$
$$= 0.$$

Similarly:

$$\frac{d}{dt}\left(z_2^2 + z_4^2\right) = 0.$$

Thus $[z_1, \cdots, z_4]'$ is bounded. Thus the function

$$V(z(t)) = -\beta\,[z_3(t) + z_4(t)] + \frac{z_5^2(t)}{2} \qquad (47)$$

is bounded from below. Further, there holds:

$$\dot{V}(z) = -\beta\,(\dot{z}_3 + \dot{z}_4) + z_5\dot{z}_5$$
$$= -\beta\,(-z_1 z_5 + \beta\alpha z_1^2 + z_2 z_5 + \beta\alpha z_2^2) - \beta z_5\,(z_1 - z_2)$$
$$= -\beta^2\alpha\,(z_1^2 + z_2^2) \leq 0. \qquad (48)$$

Thus $V(z)$ and hence $z_5(t)$ is bounded. Consequently, z is in a compact set. Thus from Lasalle's Theorem, and (48) asymptotically z converges to the trajectory corresponding to $\dot{V}(z) \equiv 0$, i.e. to (45) and (46). ∎

This brings us to our main theorem demonstrating global convergence.

Theorem 5.1. Consider (2) for $i \in \{1, 2\}$, (10) and (11) with $\beta, \alpha > 0$. Then for some ϕ, for integers m and n, $[\theta_1(t), \theta_2(t), \omega_1(t), \omega_2(t)]'$ converges uniformly asymptotically to

$$\begin{bmatrix} \dfrac{(m+n)\,\pi}{2\alpha}t + \phi + \dfrac{(m-n)\,\pi}{2} \\[2mm] \dfrac{(m+n)\,\pi}{2\alpha}t + \phi \\[2mm] \dfrac{(m+n)\,\pi}{2\alpha} \\[2mm] \dfrac{(m+n)\,\pi}{2\alpha} \end{bmatrix}.$$

Further for almost all initial conditions, m and n are even.

Proof: Under (30)-(34), (40)-(44) hold. Thus from Lemma 5.1, uniformly asymptotically, (12) and (13) hold. Then the result follows from Theorems 4.1 and 4.2. ∎

Thus global consensus involving both phase and frequency lock is indeed achieved. Further in view of Theorem 4.2 this consensus ensues at an exponential rate. Uniform asymptotic stability also guarantees robustness to noise and delay.

6. CONCLUSION

We have proposed a consensus based frequency synchronization algorithm, and proved its global convergence. We have shown that frequency lock occurs at an exponential rate, and to a frequency that is $n\pi/(2\alpha)$, α being a design parameter, and n an integer. At consensus the phase difference is $n\pi$, for the same n. Thus should n be odd, it can be detected and the *de facto* phase difference of π can be corrected.

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