Coprime Conditions for Fourier Sampling for Sparse Recovery

Hema K. Achanta, Sampurna Biswas, Soura Dasgupta, Mathews Jacob,
Bhanumati N. Dasgupta and Raghuraman Mudumbai

***Abstract*—This paper considers the spark of $L \times N$ submatrices of the $N \times N$ Discrete Fourier Transform (DFT) matrix. Here a matrix has spark m if every collection of its $m-1$ columns are linearly independent. The motivation comes from such applications of compressed sensing as MRI and synthetic aperture radar, where device physics dictates the measurements to be Fourier samples of the signal. Consequently the observation matrix comprises certain rows of the DFT matrix. To recover an arbitrary k-sparse signal, the spark of the observation matrix must exceed $2k+1$. The technical question addressed in this paper is how to choose the rows of the DFT matrix so that its spark equals the maximum possible value $L+1$. We expose certain coprimeness conditions that guarantee such a property.**

***Index Terms*—Coprime sensing, full spark, compressed sensing, vanishing sums.**

I. Introduction

Moitivated by recent work on coprime sampling, [1], this paper exposes certain coprimeness conditions that permit sparse recovery from Fourier samples. We note that the recovery of sparse signals from their under-sampled linear measurements has received considerable attention in the last few years. This approach, which is referred to as compressed sensing (CS), has many applications, including MRI [2]–[4] and synthetic aperture radar [5]–[7].

In the general CS setting, the measurement process is modeled as

$$y = Ax,$$

where $x \in \mathbb{C}^N$ is the sparse signal vector to be recovered from the observations $y \in \mathbb{C}^L$. The *observation matrix* $A \in \mathbb{C}^{L \times N}$, $L < N$ is fat. The recovery of x is formulated as a constrained optimization scheme, where the vector with the smallest support (or ℓ_0 semi-norm) that satisfies the data consistency is estimated; this scheme is often referred to as ℓ_0 recovery. The necessary and sufficient conditions to recover an arbitrary k-sparse vector using ℓ_0 recovery is now well-known to be $\textbf{spark}(A) > 2k$, [8], [9]. The spark of A is the smallest number of linearly dependent columns in A [8], [9]. Since the ℓ_0 recovery scheme involves a combinatorial search, it is computationally infeasible for large vectors. The seminal works of Candes and Tao show that the ℓ_0 recovery scheme can be replaced by a convex ℓ_1 optimization scheme with further restrictions [8], [10], [11]

In practical applications, the acquisition of each measurement comes with a penalty (e.g. acquisition time in MRI). Thus it is desirable to recover the sparse signal from as few measurements as possible. This translates to designing a measurement matrix with the highest possible spark. Since the computation of the spark of a large matrix is intractable, most of the earlier work on designing high spark matrices were based on the lower bound $\mathrm{spark}(\mathbf{A}) > 1/\mathrm{M}$, where M is the maximum off-diagonal entry of $\mathbf{A^H A}$; it is often termed as the mutual coherence of $\mathbf{A}$ [8]. Equiangular tight frame matrices are known to attain the lower bound on mutual coherence, which is known as the Welch bound [12], [13].

In many of these applications (e.g MRI, synthetic aperture radar), device physics mandates that the measurements represent Fourier samples of the signals. It is difficult to acquire the measurements using an arbitrary observation matrix. Many of the random/deterministic constructions that are available in the general setting cannot be easily translated to the Fourier sampling setting. While the common practice in CS-MRI is to randomly sample the Fourier transform using a variable density sampling pattern, there are no theoretical guarantees on the recovery using these matrices. Recently, some researchers have shown that the partial Fourier matrices, obtained by picking the rows of the Discrete Fourier Transfrom (DFT) matrix, according to a particular difference set attains the Welch bound [14], [15].

In this paper we ask the following question: Which L rows of an $N \times N$ DFT matrix yield a submatrix having the highest spark, $L+1$? An $L \times N$ matrix, , $L < N$ with spark $L+1$ is called *full spark*. It is well known that L-consecutive rows of the DFT matrix have this property. Further [16] shows that should L rows indexed by the set $\mathcal{M}$ have spark $L+1$, then so do those indexed by (a) a translation, (b) or the complement of $\mathcal{M}$, or (c) rows indexed by $M\mathcal{M}$ if M is coprime with N. Further, [16] also provides a necessary condition on $\mathcal{M}$ under which full spark obtains. It disproves the sufficiency of this condition through a counterexample.

After summarizing the results of [16] in Section II, we provide two general results in Section III. First we strenghen the implications of the coprimeness condition in (c). We then turn to the counterexample in [16], a feature of which is that it involves consecutive rows of the DFT matrix with

This work is in part supported by US NSF grants EPS-1101284, ECCS-1150801, CNS-1329657, CCF-1302456, CCF-0844812, CCF-1116067, NIH 1R21HL109710-01A1, ACS RSG-11-267-01-CCE, and ONR grant N00014-13-1-0202. All but the fifth author are with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242. {hemakumari-achanta,mathews-jacob}@uiowa.edu, sampurnakgp@gmail.com, {dasgupta,rmudumbai,} @engineering.uiowa.edu. Phone 319-335-5200. Fax 319-335-6028. The fifth author is with the Department of Mathematics, University of Iowa, Iowa City, IA 52242 bhanumati@msn.com.

precisely one intermediate row missing. Because of (a-c), the full spark nature of such a set yields several others with the same property. We consider this general setting and tie full spark to the notion of *vanishing sums* of N-th roots of unity, [17]. We show that the counterexample in [16] is consistent with these results and relate our conditions to the nonnegative integer combinations of the prime factors of N. Section IV strengthens these latter results in the case where N is a product of two primes.

II. Related Work

We begin with some notation. Throughout, our observation matrices are submatrices of the $N \times N$ DFT matrix W_N. Specifically, with rows and column indices taking values from the set $\mathbb{Z}_N = \{0, 1, \cdots, N-1\}$, the il-th element of W_N is $e^{\frac{j2\pi il}{N}}$. These are of course N-th roots of unity. We will assume that the observation matrix $A(\mathcal{M}, N)$ is obtained by retaining the rows of W_N that are indexed by $\mathcal{M} \subset \mathbb{Z}_N$. The key technical issue addressed in this paper is: *What conditions on $\mathcal{M}$ and N ensure that $A(\mathcal{M}, N)$ has full spark?*. In this section we summarize some key known results related to this question.

Suppose for some i and l, $\mathcal{M} = \{i, i+1, \cdots, i+l-1\}$ i.e. contains consecutive elements of $\mathbb{Z}_N$. Define

$$z_n = e^{j\frac{2\pi n}{N}} \tag{II.1}$$

Then the matrix comprising any l columns of $A(\mathcal{M}, N)$, indexed by the integers $i_1, \cdots, i_l$ can be expressed as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_{i_1} & z_{i_2} & \cdots & z_{i_l} \\ \vdots & \vdots & \vdots & \vdots \\ z_{i_1}^{l-1} & z_{i_2}^{l-1} & \cdots & z_{i_l}^{l-1} \end{bmatrix} \textbf{diag } \{z_{i_1}^i, z_{i_2}^i, \cdots, z_{i_l}^i\}.$$

and being a product of a Vandermonde and a nonsingular diagonal matrix, is thus nosingular for distinct z_{i_n}. Consequently, such an $A(\mathcal{M}, N)$ has full spark.

A less obvious result can be traced back to Chebotarëv in the early 20th century, (see [18]).

Theorem 2.1: Suppose N is prime. Then for all $\mathcal{M} \subset \mathbb{Z}_N$, $A(\mathcal{M}, N)$ has full spark.

The most sophisticated results are in [16] and provide a springboard for the results of this paper. As with Theorem 2.1 these results expose the role of prime factors of N and their relation to the set $\mathcal{M}$. The first is a set of necessary conditions.

Theorem 2.2: Suppose for some $\mathcal{M} \subset \mathbb{Z}_N$, $A(\mathcal{M}, N)$ has full spark. Then so does:

(i) $A((\mathcal{M}+i) \mod N, N)$ for all $i \in \mathbb{Z}_N$, i.e. the full spark condition is preserved under all translations.
(ii) $A(M\mathcal{M}, N)$ for all M that is coprime with N.
(iii) $A(\mathbb{Z}_N \setminus \mathcal{M}, N)$.

This theorem permits one to build entire classes of $\mathcal{M}_i$ for which $A(\mathcal{M}_i, N)$ is full spark, from any $\mathcal{M}$ for which $A(\mathcal{M}, N)$ is full spark.

The next result of note from [16] requires a definition. Observe a divisor d of N, partitions $\mathbb{Z}_N$ into d *cosets*, the i-th coset being defined as

$$\mathcal{C}_i(d, N) = \{\, l \in \mathbb{Z}_N |\, l \mod d = i \,\}, \quad i \in \{0, 1 \cdots, d-1\}. \tag{II.2}$$

We say that $\mathcal{M}$ is *uniformly distrbuted over the divisor d* if for each i the cardinality of $\mathcal{C}_i(d, N) \bigcap \mathcal{M}$ is either

$$\left\lceil \frac{|\mathcal{M}|}{d} \right\rceil \text{ or } \left\lfloor \frac{|\mathcal{M}|}{d} \right\rfloor.$$

Then [16] proves the following remarkable theorem.

Theorem 2.3: The matrix $A(\mathcal{M}, N)$ has full spark only if $\mathcal{M}$ is uniformly distributed over all divisors of N. If N is a prime power then $A(\mathcal{M}, N)$ has full spark iff $\mathcal{M}$ is uniformly distributed over all divisors of N.

Furthermore, [16] disproves the conjecture that $\mathcal{M}$ being uniformly distributed over all divisors of N suffices for $A(\mathcal{M}, N)$ to have full spark, regardless of whether N is a prime power, through the following counterexample.

Example 2.1: Consider $N = 10$, $\mathcal{M} = \{0, 1, 3, 4\}$. This $\mathcal{M}$ is uniformly distributed over 2 and 5. Yet the columns of $A(\mathcal{M}, 10)$ indexed by the set $\{0, 1, 2, 6\}$ are linearly dependent.

III. Some General Results

In this section we provide two types of results. The first extends a consequence of (ii) of Theorem 2.2 using the following Lemma which can be found in [19].

Lemma 3.1: Consider integers $1 \le M < N$. Then there exists $1 \le n < N$ such that N divides Mn iff M and N are not coprime.

Using this lemma we now prove the following theorem.

Theorem 3.1: Suppose for positive integers M, K, $\mathcal{M} = \{i, i+M, i+2M, \cdots, i+(K-1)M\}$, with $i+(K-1)M < N$. Then $A(\mathcal{M}, N)$ has full spark iff M and N are coprime. Further if $A(\mathcal{M}, N)$ does not have full spark, then $\textbf{spark}(A) = 2$.

Proof: Sufficiency follows from the facts that consecutive rows of W_N have full spark, and (i) and (ii) of Theorem 2.2.

With $n \in \{0, \cdots, N-1\}$, under (II.1), the l-th column of $A(\mathcal{M}, N))$ is: $a_l = z_l^i \begin{bmatrix} 1 & z_l^M & \cdots & z_l^{(k-1)M} \end{bmatrix}^\top$ Thus with

$b_l = \begin{bmatrix} 1 & z_l^M & \cdots & z_l^{(k-1)M} \end{bmatrix}^\top$ and $B = \begin{bmatrix} b_0 & b_1 & \cdots & b_{N-1} \end{bmatrix}$

there holds: $A(\mathcal{M}, N) = B\textbf{diag } \{z_l^i\}_{l=0}^{N-1}$. Thus $\textbf{spark}(A(\mathcal{M}, N)) = \textbf{spark}(B)$. Now the Vandermonde structure of B ensures that B has full spark only if for all $0 \le p < q < N$

$$z_p^M \neq z_q^M. \tag{III.1}$$

From (II.1) this is equivalent to the nonexistence of an integer l such that

$$\begin{aligned} & Mq = Mp + Nl \\ \Leftrightarrow \quad & M(q-p) = Nl. \end{aligned}$$

As $0 < n = q - p < N$, the result follows from Lemma 3.1 and the fact that the violation of (III.1) implies $\textbf{spark}(A(\mathcal{M}, N)) = 2$. ■

One notes that this is yet another connection between coprimeness and Fourier sampling that results in full spark observation matrices.

We next turn to what we regard as the main results of this paper. It is evident from Theorem 2.2 that using the fact that with $\mathcal{M} = \{0, 1, \cdots, K\}$, the full spark nature of $A(\mathcal{M}, N)$ permits the construction of a plethora of subsets of $\mathbb{Z}_N$ that yield the full spark property. The hallmark of $\mathcal{M} = \{0, 1, \cdots, K\}$ is that consecutive rows of W_N comprise the observation matrix. The rest of the paper asks: What if a *single frame* from $\{0, 1, \cdots, K\}$ is missing? What conditions guarantee full spark observation matrices? These sufficient conditions, together with Theorem 2.2 then generate a rich class of further row indices that guarantee full spark observations. On a related note, we observe that Example 2.1 also has a missing frame. Our results directly explain why it lacks full spark.

We directly relate the lack of full spark to the notion of *vanishing sums of roots of unity*, [17]. The N-th roots of unity, $\{s_i\}_{i=1}^L$, s_i not necessarily distinct, *form a vanishing sum* if

$$\sum_{i=1}^{L} s_i = 0, \tag{III.2}$$

We emphasize that in $\{s_i\}_{i=1}^L$, s_k may equal s_l, even if $k \neq l$. The Lemma below, from [17], provides a necessary condition on L for $\{s_i\}_{i=1}^L$ to form a vanishing sum. The lemma refers to *nonnegative integer combination* of integers p_i: r is a nonnegative integer combination of integers p_i if there exist nonnegative integers n_i such that

$$r = \sum_i n_i p_i.$$

Lemma 3.2: The possibly nondistinct N-th roots of unity $\{s_i\}_{i=1}^L$ form a vanishing sum only if L is a nonegative integer combination of the prime factors of N. Further, should L be a nonegative integer combination of the prime factors of N, then there is always a possibly nondistinct collection of N-th roots of unity $\{s_i\}_{i=1}^L$, that form a vanishing sum.

As will be evident in the sequel, there emerges a new condition for full spark Fourier submatrices, that involves the integer combination of the prime factors of N. Towards such a result we first provide a fairly general theorem concerning the setting where a solitary row is missing from the index set $\{0, 1, \cdots, K\}$. The theorem refers to the sum of m-products of a set of complex numbers. This is the sum of the products of the elements belonging to all subsets of the set with cardinality m. For example, the sum of 2-products of $\{a_1, a_2, a_3\}$ is $a_1a_2 + a_1a_3 + a_2a_3$.

Theorem 3.2: For integers $1 \leq n < K < N$ and $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{n\}$, $A(\mathcal{M}, N)$ does not have full spark iff there exist K distinct N-th roots of unity whose n-products form a vanishing sum.

Proof: Since it has K rows, $A(\mathcal{M}, N)$ does not have full spark iff it has K distinct columns that are linearly dependent. Suppose these columns are indexed from the set $\{n_1, \cdots, n_k\} \subset \mathbb{Z}_N$. Under (II.1), l-th of these columns comprises the powers $z_{n_l}^i$, $i \in \{0, 1, \cdots, K\} \setminus \{n\}$. Thus, their linear dependence is equivalent to the existence of a nonzero polynomial, $\theta(z)$ of degree K with the coefficient of z^n zero, whose roots are z_{n_l} for each $l \in \{1, \cdots, K\}$. The result follows from theeasily verified fact that the n-th coefficient of such a nonzero polynomial is, to within a sign, the sum of the n-products of its roots. ■

We now revisit Example 2.1 in which $N = 10$, $K = 4$, and $n = 2$. With $l \in \{0, 1, 2, 6\}$, it is readily checked that the sum of the six 2-products of the z_l is indeed zero.

In view of Lemma 3.2, and the pair of facts that the n-products of K distinct N-th roots of unity are $\binom{K}{n}$ in number, and are themseleves N-th roots of unity, the following sufficient condition is then immediate.

Theorem 3.3: For integers $1 \leq n < K < N$ and $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{n\}$, $A(\mathcal{M}, N)$ has full spark if $\binom{K}{n}$ is not a nonnegative integer combination of the prime factors of N.

Again with $N = 10$, $K = 4$, and $n = 2$, Example 2.1 violates the sufficient condition in Theorem 3.3 as $6 = \binom{4}{2}$ is a positive integer multiple of 2, a prime factor of 10. This Theorem again brings into sharp relief the role played by the prime factors of N.

Finally, we show that should for $1 \leq n < K < N$, and $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{n\}$, $A(\mathcal{M}, N)$ not have full spark, then in fact it has spark even lower than K. To this end we need a lemma.

Lemma 3.3: Suppose a nonzero polynomial with degree K has all its roots on the unit circle. Suppose also the coefficient of power of z^n, $0 < n < K$, in this polynomial is zero. Then so is the coefficient of power of z^{K-n}.

Proof: Suppose $s_1, \cdots, s_K$ are the roots of this polynomial. The coefficient of z^n is the sum of all n-products of the s_i. Thus this coefficient is zero iff the sum of all n-products of the s_i is zero. Thus this sum divided by $\prod_{i=1}^K s_i$ is also zero. As s_i are on the unit circle $1/s_i = s_i^*$. Thus this ratio is the conjugate of the sum of all the $K - n$ products of the s_i. The result follows. ■

Using this theorem and the proof technique of Theorem 3.2 the following theorem obtains.

Theorem 3.4: For integers $1 \leq n < K < N$ and $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{n\}$, suppose $A(\mathcal{M}, N)$ does not have full spark. Then with $\mathcal{M}_1 = \{0, 1, \cdots, K\} \setminus \{n, K - n\}$, $A(\mathcal{M}_1, N)$ does not have full spark either.

IV. Specialization to the case where N is a product of two primes

Theorem 3.2 characterizes conditions under which the full spark condition is satisfied when a single row is excluded from K consecutive rows of the DFT matrix, and links it to vanishing sums. Theorem 3.3 provides a sufficient condition on, N, K and n for the full spark condition to hold. Between them, these two theorems do not however provide a necessary and sufficient condition on the integers N, K and n.

To partially address this gap we consider the special case where N is the product of two distinct prime factors, i.e. with

p and q distinct primes

$$N = pq. \tag{IV.1}$$

In this case [17] provides an easy characterization of the N-th roots of unity that form a vanishing sum. Specifically, consider the two sets:

$$Z_q = \{\, lq |\, l \in \{0, \cdots, p-1\}\} \tag{IV.2}$$

and

$$Z_p = \{\, lp |\, l \in \{0, \cdots, q-1\}\}\,. \tag{IV.3}$$

Observe, for any a, b, (see (II.1)),

$$a \sum_{i \in Z_q} z_i = 0 \text{ and } b \sum_{i \in Z_p} z_i = 0$$

Then the sets of possibly nondistinct $\{z_{i_l}\}_{l=1}^{L}$ that form a vanishing sum obey two conditions:

(A) For some nonnegative integers α and β the total number L of these z_{i_l} is $\alpha p + \beta q$.

(B) The set $\{1, 2, \cdots, L\}$ can be partitioned into α sets $\{S_{pi}\}_{i=1}^{\alpha}$ and β sets $\{S_{qi}\}_{i=1}^{\beta}$ such that for some integer r_i, $S_{pi} = r_i + Z_p$ and for some integer t_i, $S_{qi} = t_i + Z_q$.

In view of Theorems 3.2 and 3.3, this almost immediately yields the following result:

Theorem 4.1: For distinct primes p and q, integers $1 < K < N = pq$, and $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{1\}$ or $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{K-1\}$, $A(\mathcal{M}, N)$ does not have full spark iff K is not a nonnegative integer combination of p and q.

Observe this theorem exploits the fact that if K numbers on the unit circle sum to zero, then so do their $K-1$ products. Note also that a somewhat surprising coprimeness condition has emerged involving the nonegative integer combinations of the prime factors of N. Further, for N a product of two primes, the condition in Theorem 3.3 is both necessary and sufficient for $n = 1$ and $n = K-1$.

What about other values of n in the notation of Theorem 3.3? While $K < N$ distinct roots of unity can be chosen arbitrarily, there n sums in general cannot be. Thus even if $\binom{K}{n}$ is a nonnegative integer combination of p and q, the n-products of K, N-th roots of unity need not partition in a manner mandated by (A) and (B) above.

A somewhat involved set of arguments, which we omit here due to space constraints, can be used to prove a less conservative version of Theorem 3.3.

Theorem 4.2: Suppose p and q are distinct primes, $N = pq$, $1 \leq n < K < N$ and $\mathcal{M} = \{0, 1, \cdots, K\} \setminus \{n\}$. Then $A(\mathcal{M}, N)$ has full spark if K is not a nonnegative integer combination of the prime factors of p and q.

This theorem is less conservative than Theorem 3.3 as $\binom{K}{n}$ may be a nonnegative integer combination of p and q, even if K is not.

V. Conclusion

We have derived certain coprimeness conditions that guarantee that full spark matrices result from appropriate Fourier sampling. In the first result we show that the rows of W_N chosen from the index set $\{i+nM, i+(n+1)M, \cdots, i+LM\}$, yields a full spark matrix iff M and N are coprime. We then turn to the case where the index set comprises $\{0, \cdots, K\}\setminus\{i\}$ where $1 < i < K$. Because of Theorem 2.2, the full spark nature of such a set yields several others with the same property. We show that full spark is equivalent to the existence of K, N-th roots of unity sum of whose i-products is zero. A sufficient condition is that $\binom{K}{i}$ not be a nonnegative integer combination of the prime factors of N. We strengthen this result for the special case when N is the product of two primes by showing that full spark results if K is not a nonnegative integer combination of p and q.

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