

# A Semidefinite Programming Approach to Source Localization in Wireless Sensor Networks

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**Abstract**—We propose a novel approach to the source localization and tracking problem in wireless sensor networks. By applying minimax approximation and semidefinite relaxation, we transform the traditionally nonlinear and nonconvex problem into convex optimization problems for two different source localization models involving measured distance and received signal strength. Based on the problem transformation, we develop a fast low-complexity semidefinite programming (SDP) algorithm for two different source localization models. Our algorithm can either be used to estimate the source location or be used to initialize the original nonconvex maximum likelihood algorithm.

**Index Terms**—Maximum likelihood estimation, semidefinite programming, source localization, wireless sensor network.

## I. INTRODUCTION

THE problem of locating a signal source using a wireless network of sensors has been addressed in several past papers [1], [2] and references therein. One solution to this problem applies weighted least squares (WLS) estimation [3]–[5] that can be quite effective. However, the WLS approach operates with a cost function that is nonconvex and subject to local minima, thus requiring either smart initialization or special conditions to ensure global convergence. To remedy the local convergence problem, an algorithm based on the concept of *projection onto convex sets (POCS)* has recently been proposed in [6] and [7]. The POCS algorithm is very effective in overcoming the local convergence problem when the source resides within the convex hull of the sensor nodes. Nevertheless, the POCS algorithm does suffer from poor performance when trying to locate a source node not inside the convex hull. In this work, we present a new approach to tackle the local convergence problem that the WLS approach faces and the problem with source outside the convex hull that the POCS algorithm faces. By transforming the source localization problem into a convex optimization problem using minimax

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approximation and semidefinite relaxation, we can find the global minimum of the relaxed problem and do not need to impose coverage conditions.

The application of semidefinite programming (SDP) in sensor localization with noisy distance measurement and angle of arrival has been studied by previous researchers, e.g., [8] and references therein. Nevertheless, instead of minimizing the  $l^1$  measurement error, this work derives the SDP from the maximum likelihood estimation and the minimax approximation. Another important feature of our algorithm is that it can be applied to two sensor measurement models. In other words, the SDP can be applied to the problem of measured distance with additive white Gaussian noise and the problem of received signal strength with lognormal fading.

Finally, the newly proposed SDP algorithm can be efficiently implemented via a standard numerical optimization toolbox such as SeDuMi [9]. The implementation also has very fast convergence. Indeed, the computation complexity is polynomial-time [10], with the worst-case scenario of  $O(m^{6.5})$ , where  $m$  is the dimension of space in which sensors are placed. This feature makes it appealing for resource-limited applications and for tracking targets in real time.

## II. PROBLEM STATEMENT

We now describe our problem formulation and basic assumptions. The known sensor positions are a set of  $m$ -dimensional vectors  $\mathbf{x}_1, \dots, \mathbf{x}_N$ . The signal source location is an unknown vector  $\mathbf{y}$ . A source location estimate  $\hat{\mathbf{y}}$  is determined from sensor measurements. In this letter, we only consider single measurement from each sensor node. Nevertheless, our results are readily applicable to cases involving multiple measurements. We consider the following two measurement models. Note that the major difference between the two models is the different assumption of noise, and the scenarios can be found in the referred literature.

### A. Distance Measurement Model

In this common model [5], [11], the distance  $d_i$  from source to each sensor node is measured, i.e.,

$$d_i^2 = \|\mathbf{x}_i - \mathbf{y}\|^2 + n_i, \quad i = 1, \dots, N \quad (1)$$

where the noise  $n_i$  is i.i.d. Gaussian with zero mean and variance  $\sigma^2$ . The conditional probability density is

$$p(d_1^2, d_2^2, \dots, d_N^2 | \mathbf{y}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\sum_{i=1}^N \frac{(d_i^2 - \|\mathbf{x}_i - \mathbf{y}\|^2)^2}{2\sigma^2}}$$

and the maximum likelihood estimate (MLE) of  $\mathbf{y}$  is

$$\hat{\mathbf{y}} = \arg \min_{\mathbf{y}} \sum_{i=1}^N (d_i^2 - \|\mathbf{x}_i - \mathbf{y}\|^2)^2. \quad (2)$$

This model is less realistic in that  $n_i$  cannot be truly Gaussian as  $d_i^2$  is nonnegative.

### B. Signal Strength Measurement Model

If the source at  $\mathbf{y}$  emits a signal of power  $P$ , the sensor at  $\mathbf{x}_i$  can receive a signal power  $s_i$

$$s_i = P \|\mathbf{x}_i - \mathbf{y}\|^{-\beta} \quad (3)$$

where  $\beta$  is the path loss coefficient [12]. Under lognormal fading, the received signal strength (RSS) in dB at a sensor can be modeled as [7], [12], [13]

$$10 \log s_i = 10 \log P - 10\beta \log(\|\mathbf{x}_i - \mathbf{y}\|) + n_i \quad (4)$$

where the noise  $n_i$  is assumed i.i.d. Gaussian with zero mean and variance  $\sigma^2$ . The conditional probability density is

$$p(\ln s_1, \ln s_2, \dots, \ln s_N | \mathbf{y}) = \frac{1}{(2\pi(\ln 10/10 \cdot \sigma)^2)^{N/2}} e^{-\frac{\sum_{i=1}^N \left( \ln s_i - \ln \left( \frac{P}{\|\mathbf{x}_i - \mathbf{y}\|^\beta} \right) \right)^2}{2(\ln 10/10 \cdot \sigma)^2}}$$

and the MLE of  $\mathbf{y}$  can be obtained as

$$\hat{\mathbf{y}} = \arg \min_{\mathbf{y}} \sum_{i=1}^N \left( \ln s_i - \ln \left( \frac{P}{\|\mathbf{x}_i - \mathbf{y}\|^\beta} \right) \right)^2. \quad (5)$$

## III. OUR NEW EFFICIENT ALGORITHMS

### A. Semidefinite Relaxation With Distance Measurement

The optimization problem of (5) involves high-order power of  $\mathbf{y}$  and is highly nonlinear and nonconvex. To simplify, we propose a minimax approximation

$$\mathbf{y}^* = \arg \min_{\mathbf{y}} \max_{i=1,2,\dots,N} |d_i^2 - \|\mathbf{x}_i - \mathbf{y}\|^2|. \quad (6)$$

This approximation is supported by the so-called equivalence between  $l^2$  and  $l^\infty$  vector norms.

To proceed, define an  $(m+1) \times 1$  vector  $\bar{\mathbf{y}} = [\mathbf{y}^T \ 1]^T$  and

$$\mathbf{X}_i = \begin{bmatrix} \mathbf{I} & -\mathbf{x}_i \\ -\mathbf{x}_i^T & \mathbf{x}_i^T \mathbf{x}_i \end{bmatrix}.$$

Then (6) can be written in an equivalent form

$$\begin{aligned} & \min t \\ \text{s.t. } & \bar{\mathbf{y}}(m+1) = 1, \quad -t < \bar{\mathbf{y}}^T \mathbf{X}_i \bar{\mathbf{y}} - d_i^2 < t, \quad i = 1, \dots, N \end{aligned}$$

By denoting  $\mathbf{Y} = \bar{\mathbf{y}}\bar{\mathbf{y}}^T$  and applying semidefinite relaxation [10], [14], this problem can be reformulated into

$$\begin{aligned} & \min t \\ \text{s.t. } & -t < \text{Trace}(\mathbf{X}_i \mathbf{Y}) - d_i^2 < t \\ & i = 1, 2, \dots, N \\ & \mathbf{Y} \succeq 0, \quad \mathbf{Y}(m+1, m+1) = 1 \end{aligned} \quad (7)$$

where  $\mathbf{Y} \succeq 0$  denotes (symmetric) positive semidefinite. Equation (7) is a convex optimization problem. Its global optimal solution can be found using modern SDP solvers such as SeDuMi [9] that applies the interior point method.

### B. Semidefinite Relaxation for Received Signal Strength

We first reformulate the localization problem of (5) into a minimax problem

$$\begin{aligned} \hat{\mathbf{y}} &= \arg \min_{\mathbf{y}} \max_{i=1,2,\dots,N} \left| \ln s_i - \ln \left( \frac{P}{\|\mathbf{x}_i - \mathbf{y}\|^\beta} \right) \right| \\ &= \arg \min_{\mathbf{y}} \max_{i=1,2,\dots,N} \left| \ln \left( \left( \frac{s_i}{P} \right)^{2/\beta} \|\mathbf{x}_i - \mathbf{y}\|^2 \right) \right|. \end{aligned} \quad (8)$$

We can show that this new optimization problem is less sensitive to the path loss parameter  $\beta$ . To do so, first note that from (4), we have

$$s_i = \frac{P \cdot 10^{n_i/10}}{\|\mathbf{x}_i - \mathbf{y}\|^\beta}.$$

The minimum of the cost function in (8) assumes the value

$$\begin{aligned} J(\hat{\mathbf{y}}) &= \left| \ln \left( \left( \frac{s_i}{P} \right)^{2/\beta} \|\mathbf{x}_i - \hat{\mathbf{y}}\|^2 \right) \right| \\ &= \left| \frac{n_i}{5\beta} \ln 10 - 2 \ln \|\mathbf{x}_i - \mathbf{y}\| + 2 \ln \|\mathbf{x}_i - \hat{\mathbf{y}}\| \right| \end{aligned}$$

for a certain  $i$ . The estimate  $\hat{\mathbf{y}}$  demonstrates insensitivity to the path loss parameter in that for low noise  $n_i$ , small perturbation of  $\beta$  has little effect on the estimate of  $\hat{\mathbf{y}}$ . Particularly when in noise-free cases, the minimum cost is independent of  $\beta$ . This is not the case for algorithms such as POCS in which  $\beta$  affects the radius of a convex set. Empirically,  $\beta$  only represents the average path-loss effect. In a practical environment, different sensors may have significantly different values of  $\beta$ , which is a well-recognized drawback of applying the RSS signal model when the path loss parameters vary. Hence, algorithms that are insensitive to  $\beta$  are generally desirable.

To find an efficient solution, first denote

$$q_i = \left( \frac{s_i}{P} \right)^{2/\beta}.$$

We transform (8) into its equivalent form

$$\begin{aligned} & \min t \\ \text{s.t. } & \bar{\mathbf{y}}(m+1) = 1 \\ & -t < \ln(q_i \|\mathbf{x}_i - \mathbf{y}\|^2) < t, \quad i = 1, 2, \dots, N. \end{aligned}$$

Using the same relaxation technique as in the previous section, we have the new problem as

$$\begin{aligned} & \min t \\ \text{s.t. } & -t < \ln(q_i \cdot \text{Trace}(\chi_i \mathbf{Y})) < t, \quad i = 1, 2, \dots, N \quad (9) \\ & \mathbf{Y} \succeq 0, \quad \mathbf{Y}(m+1, m+1) = 1. \end{aligned}$$

To make first constraint in (9) more tractable, additional approximations can be applied. First observe that  $\ln x \leq x - 1$ . Thus, a necessary condition for

$$-t < \ln(q_i \text{Trace}(\chi_i \mathbf{Y}))$$

is

$$-t < q_i \text{Trace}(\chi_i \mathbf{Y}) - 1.$$

We therefore have two necessary conditions

$$\begin{aligned} 1 - t &< q_i \text{Trace}\{\chi_i \mathbf{Y}\} \\ 0 &< \text{Trace}\{\chi_i \mathbf{Y}\}. \end{aligned}$$

Because both  $\mathbf{Y}$  and  $\chi_i$  are symmetric positive semidefinite

$$\text{Trace}\{\chi_i \mathbf{Y}\} > 0$$

is guaranteed and becomes redundant. Now consider the right-hand side of the first constraint

$$\ln(q_i \text{Trace}\{\chi_i \mathbf{Y}\}) < t.$$

We have a sufficient condition

$$\ln(q_i \text{Trace}\{\chi_i \mathbf{Y}\}) < q_i \text{Trace}\{\chi_i \mathbf{Y}\} - 1 < t.$$

To summarize, we transform (9) into an approximation

$$\begin{aligned} & \min t \\ \text{s.t. } & -t < q_i \cdot \text{Trace}(\chi_i \mathbf{Y}) - 1 < t, \quad i = 1, 2, \dots, N \\ & \mathbf{Y} \succeq 0, \quad \mathbf{Y}(m+1, m+1) = 1. \end{aligned} \quad (10)$$

The optimization problem in (10) can now be solved using standard SDP tools. Note that, in general, semidefinite relaxation increases the problem size, and some postprocessing techniques are required to convert the SD relaxation solution  $\mathbf{Y}$  into an approximate solution of the original optimization problem in terms of  $\hat{\mathbf{y}}$ . Such standard techniques have been routinely applied in signal processing and communications [14], [15].

#### IV. SIMULATIONS

We provide two examples for performance comparison.

*Example 1:* Consider the distance measurement model. Sensors are located in 2-D at  $\mathbf{x}_1 = [2, 2]^T$ ,  $\mathbf{x}_2 = [2, 6]^T$ ,  $\mathbf{x}_3 = [6, 2]^T$ . The source is first placed at  $\mathbf{y} = [0, 0]^T$  (outside the convex hull formed by the sensors) and later placed at  $\mathbf{y} = [3, 3]^T$  (inside the convex hull). Note that when the source is outside the convex hull, the POCS performance is generally bad

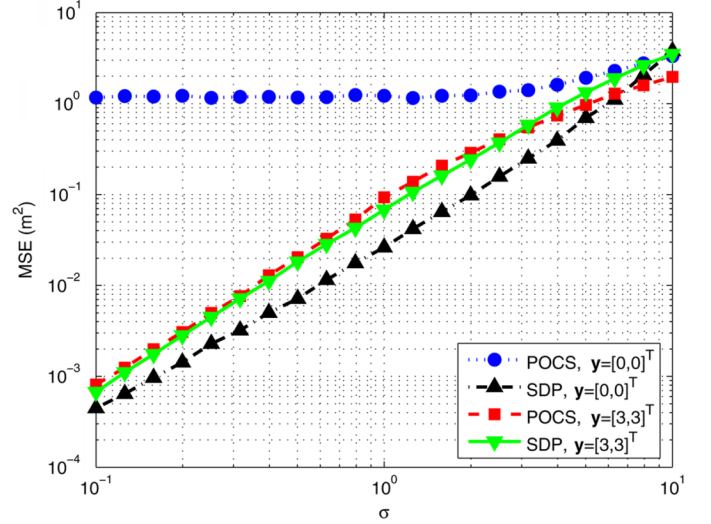


Fig. 1. The 2-D simulation results based on the measured distance model.

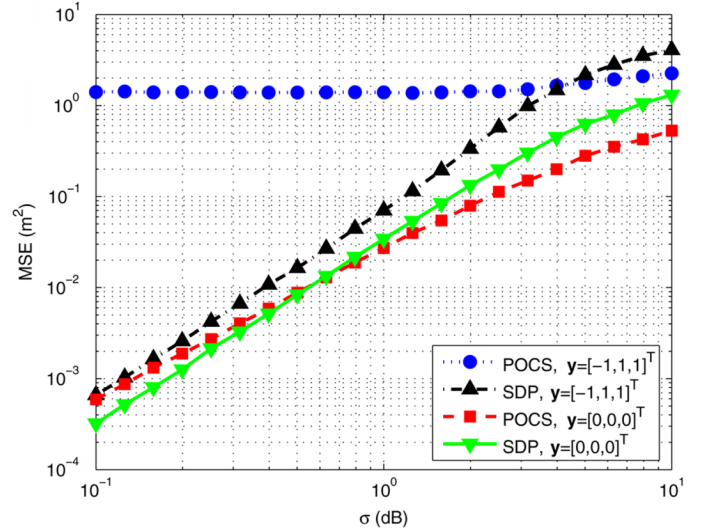


Fig. 2. The 3-D results under the received signal strength model.

because the intersection of convex sets is quite large, giving rise to many potential solutions. Also note that in this case, a spurious stationary point  $\mathbf{y}' = [6, 6]^T$  is locally stable, when solving the maximum likelihood estimation using gradient descent methods [5]. In Fig. 1, we compare the estimation results of POCS and SDP for the two different source locations. The results are given as the mean-squared error of the estimate versus standard deviation of the noise, averaged over 3000 Monte Carlo tests. Good estimation is demonstrated consistently for the SDP method, while POCS fails to locate the source consistently.

*Example 2:* We now consider the RSS model. We set  $P = 1000$ ,  $\beta = 3$ , and distribute five sensors in 3-D at  $\mathbf{x}_1 = [1, 0, 0]^T$ ,  $\mathbf{x}_2 = [0, 2, 0]^T$ ,  $\mathbf{x}_3 = [-2, -1, 0]^T$ ,  $\mathbf{x}_4 = [0, 0, 2]^T$ ,  $\mathbf{x}_5 = [0, 0, -1]^T$ . We first put the true source at  $\mathbf{y} = [-1, 1, 1]^T$  (outside the convex hull of sensors) and then at  $\mathbf{y} = [0, 0, 0]^T$  (inside the convex hull). In both cases, the SDP algorithm provides consistently good estimates while POCS works well for the second source but fails to locate the first one. The results are illustrated in Fig. 2.

## V. CONCLUSION

We propose a novel approach to the source localization problem in wireless sensor networks. Our algorithm relies on the concept of minimax approximation to the optimal maximum likelihood estimation, and it takes the form of an efficient semidefinite programming. We study two different measurement models for source localization and present well-posed solutions in both cases. We present simulation examples that demonstrate the good performance and consistency of the new SDP algorithms. Our algorithm is also less sensitive to inaccuracy in the path loss factor of the RSS model. The proposed SDP estimate can serve either as a standalone solution or as an initialization to the more complex maximum likelihood algorithms.

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