Circumnavigation Using Distance Measurements Under Slow Drift

Iman Shames, *Member, IEEE*, Soura Dasgupta, *Fellow, IEEE*, Barış Fidan, *Senior Member, IEEE*, and Brian D. O. Anderson, *Life Fellow, IEEE*

***Abstract*—Consider an agent A at an unknown location, undergoing sufficiently slow drift, and a mobile agent B that must move to the vicinity of and then circumnavigate A at a prescribed distance from A. In doing so, B can only measure its distance from A, and knows its own position in some reference frame. This paper considers this problem, which has applications to surveillance and orbit maintenance. In many of these applications it is difficult for B to directly sense the location of A, e.g. when all that B can sense is the intensity of a signal emitted by A. This intensity does, however provide a measure of the distance. We propose a nonlinear periodic continuous time control law that achieves the objective using this distance measurement. Fundamentally, a) B must exploit its motion to estimate the location of A, and b) use its best instantaneous estimate of where A resides, to move itself to achieve the circumnavigation objective. For a) we use an open loop algorithm formulated by us in an earlier paper. The key challenge tackled in this paper is to design a control law that closes the loop by marrrying the two goals. As long as the initial estimate of the source location is not coincident with the intial position of B, the algorithm is guaranteed to be exponentially convergent when A is stationary. Under the same condition, we establish that when A drifts with a sufficiently small, unknown velocity, B *globally* achieves its circumnavigation objective, to within a margin proportional to the drift velocity.**

***Index Terms*—Nonlinear periodically time varying (NLPTV) algorithm.**

I. Introduction

In surveillance and orbiting missions it is often desirable to monitor a target by circumnavigating it from a prescribed distance. In recent years this problem has been addressed in the context of autonomous agents, where an agent or a group of agents accomplish the surveillance task. Most of these studies are for the case where the position of the target is known and the agent can measure specific information about the target, such as the distance of the target or power, angle of arrival, time difference of arrival, of signals emitted by the target, etc. See [1]–[6] and references therein. However, in many situations, the assumption of knowing the position of the target is not always practical, e.g. if one wants to find and monitor the source of an electromagnetic signal at an unknown location. This paper addresses the problem where the position of the target, e.g. signal source, is unknown and the source might be undergoing a slow and potentially nontrivial drift; only one agent is involved in monitoring this target; and the only information continuously available to the agent is its own position and its distance (not relative position) from the target. Another work that considers a similar problem to this paper is [7], where the same circumnavigation problem is considered with bearing, rather than distance measurements.

Circumnavigating a target at an unknown position at first seems trivial to accomplish. One simply needs three distance measurements in $\mathbb{R}^2$ from noncollinear points, or four in $\mathbb{R}^3$ from noncoplanar points, at different time instants to estimate the position of the target, and then one can apply a simple control law to start rotating around the estimated position of the target. However, this approach has two main disadvantages. First, it will not be robust to any noise corrupting the distance measurements. Second the target may move between consecutive measurements and indeed after the final measurement. At a minimum, such a "once only estimation approach" cannot cope with sustained drift in the target. In this paper we propose a continuous time *nonlinear periodically time varying (NLPTV)* algorithm that adaptively estimates the position of the target and moves the agent to a trajectory encircling it.

To be specific, for a stationary target, this algorithm achieves the circumnavigation objective *effectively globally* and exponentially fast. By effective global convergence we mean that convergence is guaranteed as long as the initial estimate of the target location differs from the initial position of the circumnavigating agent, a condition that is easy to satisfy. When the target moves with a sufficiently small *albeit unknown and not necessarily constant* velocity, even if the *drift persists indefinitely*, the circumnavigation objective is accomplished, again effectively globally, to within an error that is proportional to the maximum target speed. A qualitative description of how small a velocity is small enough is provided.

A. Context of This Paper

Most papers for meeting distance specifications assume the knowledge of the target *position*. Since distances are nonlinear

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I. Shames is with ACCESS Linnaeus Centre, Electrical Engineering, Royal Institute of Technology (KTH), Stockholm SE-100 44, Sweden (e-mail: imansh@kth.se).

S. Dasgupta is with Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242 USA (e-mail: dasgupta@engineering.uiowa.edu).

B. Fidan is with Department of Mechanical and Mechatronics Engineering, University of Waterloo, Waterloo, ON N2L 3G1, Canada (e-mail: fidan@uwaterloo.ca).

B. D. O. Anderson is with Australian National University and National ICT Australia, Canberra, Australia (e-mail: brian.anderson@anu.edu.au).

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functions of relative positions, the resulting control laws are nonlinear and only *locally stable*, e.g. [8]. On the other hand our circumnavigation objective also involves meeting distance specifications and must thus require a nonlinear control law. In our case the nature of the required nonlinearity is compounded by the fact that only distance, as opposed to position measurements, are available to guide control action. Despite this, the control law we propose is *effectively globally stable*.

In a series of papers Cao and Morse [9]–[11] using concepts from switched adaptive control, do consider the case where an agent must move itself to a point at a pre-set distance from three sources with unknown position in the plane, using distance measurements. This is clearly a different problem to that considered here, albeit one which requires some form of dual control for its solution. At least in concept these algorithms are globally stable. They entail however, the repeated online solution of complicated optimization problems that are either nonconvex, or are reducible to eight separate convex problems, that still demand complicated computations. By contrast our algorithm is computationally simple and requires no such optimization. Additionally, unlike [9]–[11] our algorithm provably copes with a drifting target. Our method also provides a natural platform to investigate the case where a group of agents is required to take up a circular formation around a target, perhaps with specified angular spacing. Thus in a sense apart from tackling a problem that is important in its own right, this paper demonstrates the feasibility of devising computationally simple effectively globally stable robust control laws that meet distance based objectives using only distance measurements.

As noted in the foregoing other papers on circumnavigation fall into two categories. In the first the position of the target is assumed known, and relative positions are used to effect circumnavigation. In the second, bearing information is used. Both require more sophisticated sensors than those for measuring received signal strength, in turn sufficient to provide distance information. This does, however, require a callibration of the signal strength emitted by the target. While at first sight it may appear that our algorithm requires the absolute position of the agent, it does so only in the reference frame of the agent's choosing. Thus effectively our algorithm also uses relative position information, in this case indirectly provided by distance measurements.

B. Approach and Challenges

The algorithm we formulate executes two steps simultaneously. The first, called the *localization step* uses the distance measurement to generate an estimate of the target location. The second, *the control step*, treats this estimate as the true location and circumnavigates the estimated position.

The underlying philosophy is akin to certainty equivalence. Specifically, if the localization step leads to an estimate that exponentially converges to the true target location, and the control step forces the agent to exponentially meet the circumnavigation objective around the location estimate, then the overall circumnavigation objective around the true target position should be exponentially met.

The challenge of this paper is not in the localization step, which is borrowed from [12]. Rather the nontrivial novelty is in the formulation of a control step that interacts with the algorithm in [12] to achieve the overall circumnavigation objective. In particular, [12], does not *close the loop* so to speak as its algorithm has no control objectives, and is driven solely by the need to localize. As such, [12] assumes that the agent can move as it pleases. Consequently the convergence proof of the localization algorithm given in [12], is trivial, following standard adaptive systems analysis techniques, [13].

A necessary and sufficient condition for exponential localization given in [11], is a *persistent excitation (p.e.)* condition that requires the agent to move in a trajectory that is not confined to a straight line in two dimensions and to a plane in three dimensions. Further, this avoidance of collinear/coplanar motion must be *persistent* in a sense described in [12]. In part this means that the agent cannot simply head straight towards the target but must execute a richer class of motion. There is a need to reconcile the p.e. requirement with the circumnavigation objective. *Thus the first challenge of this paper is to devise a control law that forces the agent to execute a motion that satisfies the p.e. condition while still achieving the control objective*.

The second challenge is to prove convergence in this closed loop setting even when the target is stationary. In particular it is relatively easy to prove that the control law forces the agent position to exponentially circumnavigate the *estimated, as opposed to the true traget position* at the prescribed distance. It is also easy to show that *should the estimated target position be correct* then the control law does indeed force the agent trajectory to obey the p.e. condition. Clearly these two properties by themselves are not enough. The direct use of the second property has an inherent circularity in its logic, as all it states is that p.e. is obtained once the localization step has converged, *while p.e. is needed to secure this convergence in the first place. The technical challenge brought about by closing the loop is to supply the missing piece that demonstrates that the agent trajectory meets the p.e. condition even in the transient phase*.

The third challenge is to prove stability with a drifting target. As can be imagined our stability analysis has two parts. We first show that when the target is stationary, the circumnavigation objective is met exponentially. Drift is tackled using robustness considerations. However, inherent to the circumnavigation objective is the constraint that even in the stationary case only a part of the state converges exponentially to a point. The remainder converges exponentially to a trajectory that is not completely specified. This is a classic *partial stability* setting. One cannot directly appeal to standard inverse Lyapunov theorems to establish robustness to slow drift. Nor is it easy to apply any of the known techniques documented in the partial stability literature [9]. Rather, we define a *reduced state space* that permits us to invoke standard inverse Lyapunov theory. The state variables in this reduced state space comprise only those signals that converge exponentially to zero, when the target is stationary. The remaining variables appear as time varying parameters in the kernel of this reduced state space. Stability of this new state space is proven under certain conditions on these new parameters, conditions that hold regardless of convergence and slow drift. We regard the use of such a reduced state space to be a technique that can potentially be used in other partial stability problems.

In the next section the circumnavigation problem described above is formally defined. Section III, provides our algorithm. A preliminary analysis of the control laws is presented in Section IV. Section V provides a reduced state space that assists in our analysis. In Section VI we establish the exponential stability of the system when the target is stationary. The stability of the system is shown for the case where the target is undergoing a drift in Section VII. In Section VIII a method to choose one of the parameters in the control laws is presented. Simulations are in Section IX.

II. Problem Statement

In what follows we formally define the problem addressed in this paper and introduce relevant assumptions.

Problem 2.1: Consider a target at an unknown position $x(t)$ and an agent at known position $y(t)$ in $\mathbb{R}^n (n \in \{2,3\})$ at time $t \in [0,\infty)$. Knowing $y(t)$, in some reference frame chosen by the agent, a desired distance d, and the measurement

$$D(t) = \|y(t) - x(t)\| \tag{II1}$$

for each time instant $t \in [0,\infty)$, find a control law that ensures that the following hold asymptotically: (a) When x is constant, $y(t)$ circumnavigates[1] x at a distance d from x. In particular, $\dot{y}(t) \neq 0$ for all t and

$$\lim_{t\to\infty} D(t) = d. \tag{II2}$$

There is an ϵ^* and a constant K, such that whenever $\|\dot{x}(t)\| \leq \epsilon < \epsilon^*$, there holds

$$\limsup_{t\to\infty} |D(t) - d| \leq K\epsilon. \tag{II3}$$

Here as in the rest of the paper $\|.\|$ denotes the 2-norm. In the problem statement (a) ensures that when the target is stationary, $y(t)$ rotates around x at a distance d. Item (b) requires that this circumnavigation be robust to drift.

We will assume that the agent can execute any motion of the form $\dot{y}(t) = u(t)$, where $u : \mathbb{R} \to \mathbb{R}^n$ obeys for some constant C_1, and all t, $\|u(t)\| + \|\dot{u}(t)\| \leq C_1$. These ensure that the force on the agent is bounded. As noted in the introduction, our two-pronged approach to this problem is as follows. We simultaneously estimate $x(t)$ and devise a law that forces $y(t)$ to circumnavigate the estimate of $x(t)$ at a distance d from it. One can break down Problem 2.1 into the following two subproblems:

1) How can one estimate $x(t)$ from the distance measurements without explicit differentiation[2] of any signal?
2) How can one make the agent move on a trajectory that is ultimately at a distance d from the estimate of $x(t)$?

[1]We will make precise what "circumnavigate" means in the sequel. In spirit this means that the trajectory asymptotically attained by the agent lets it view the target from a sufficiently rich set of perspectives.

[2]Excluding differentiation is, at least roughly speaking, equivalent to excluding measurement of relative speed or relative velocity. Such measurements could be contemplated with a further sensor.

III. Proposed Algorithm

As noted earlier, the algorithm we enunciate comprises two steps to be simultaneously executed. Section III-A describes the localization step that is designed to estimate the location of the target from the distance measurements $D(t)$. Section III-B describes the control step that is designed to force the agent to circumnavigate the estimate provided by the localization step.

A. The Localization Step

The localization algorithm for estimating $x(t)$, from $D(t)$, is the algorithm formulated in [12]. To make the paper more self contained, beyond just stating the algorithm we also provide the intuition behind it. First observe that because of (II1), when x is a constant, one obtains

$$\frac{d}{dt}D^2(t) = 2\dot{y}^T(t)\left(y(t) - x\right) = \frac{d}{dt}\|y(t)\|^2 - 2\dot{y}^T(t)x.$$

Thus, with $\hat{x}(t)$ an estimate of x

$$\frac{1}{2}\left(\frac{d}{dt}D^2(t) - \frac{d}{dt}\|y(t)\|^2\right) + \dot{y}^T(t)\hat{x}(t) = \dot{y}^T(t)\left(\hat{x}(t) - x\right). \tag{III1}$$

Consequently, if x is constant for any $\gamma > 0$, the algorithm

$$\dot{\hat{x}}(t) = -\gamma\dot{y}(t)\left[\frac{1}{2}\frac{d}{dt}\left(D^2(t) - \|y(t)\|^2\right) + \dot{y}^T(t)\hat{x}(t)\right] \tag{III2}$$

reduces to

$$\frac{d}{dt}\left(\hat{x}(t) - x\right) = -\gamma\dot{y}(t)\dot{y}^T(t)\left(\hat{x}(t) - x\right). \tag{III3}$$

Differential equations such as (III3) have been well studied in the adaptive identification literature. In particular in (III3), $\hat{x}(t)$ converges exponentially to x, provided $\dot{y}(\cdot)$ is p.e., [13], with p.e. defined in Definition 3.1 below. In this definition, and elsewhere in the paper, for square matrices A and B, $A > (\geq) B$, designates that $A - B$ is positive (semi)definite.

Definition 3.1: [13] Consider any positive integer N and a signal $r : \mathbb{R} \to \mathbb{R}^N$. Then $r(\cdot)$ is p.e. if there exist positive α_i and T_1, such that for all t, there holds

$$\alpha_1 I \leq \int_t^{t+T_1} r(\tau)r(\tau)^\top d\tau \leq \alpha_2 I$$

Essentially this requires that $r(\cdot)$ persistently span $\mathbb{R}^N$. The upper bound is simply a boundedness assumption. The α_i and T_1 will be called p.e. parameters. As shown in [12] for $n = 2$, $\dot{y}(\cdot)$ being p.e., is equivalent to the requirement that over every time interval of length T_1, the minimum distance of $y(\cdot)$ from any straight line be larger than a number that grows with α_1, i.e. $y(\cdot)$ persistently avoids a linear trajectory. Similarly, for $n = 3$, $y(\cdot)$ must persistently avoid a planar trajectory. This accords with intuition as in $\mathbb{R}^2$, one must have distances from noncollinear points to achieve localization, just as in $\mathbb{R}^3$ one must have distances from noncoplanar points. Persistent avoidance is needed for exponential convergence.

Thus, (III1) can serve as a localization step. *However its implementation requires the differentiation of $D(t)$ rendering it impractical*. Instead, for $\alpha > 0$, [12] generates

$$\eta(t) = \dot{z}_1(t) = -\alpha z_1(t) + \frac{1}{2}D^2(t) \tag{III4}$$

$$m(t) = \dot{z}_2(t) = -\alpha z_2(t) + \frac{1}{2}y^\top(t)y(t) \tag{III5}$$

$$V(t) = \dot{z}_3(t) = -\alpha z_3(t) + y(t) \tag{III6}$$

where $z_1(0)$, and $z_2(0)$ are arbitrary scalars, and $z_3(0)$ is an arbitrary vector. Note that the generation of $\eta(t)$, $m(t)$ and $V(t)$ requires simply the measurements $D(t)$ and the knowledge of the localizing agent's own position, and can be performed without explicit differentiation.

Then the localization algorithm of [12] and indeed the localization step used in this paper is defined by (III4) to (III6) and (III7) below

$$\dot{\hat{x}}(t) = -\gamma V(t)\left(\eta(t) - m(t) + V^\top(t)\hat{x}(t)\right) \tag{III7}$$

Here $\hat{x}(t)$ denotes the estimate of $x(t)$ at time t, and $\gamma > 0$ is an adaptation gain. As argued in [12], the signals $\eta(\cdot)$, $m(\cdot)$ and $V(\cdot)$ are just filtered versions of the derivatives of $D^2(t)/2$, $\|y\|^2/2$ and $y(t)$, respectively. Further, as shown in [12], and also in Section IV, when x is a constant, one has

$$\eta(t) - m(t) + V^\top(t) = \hat{x}(t)V^\top\left(\hat{x}(t) - x\right) + e(t) \tag{III8}$$

where $e(t)$ is an exponentially decaying signal. Thus, for stationary x, to within an exponentially decaying signal, (III7) has the form of (III2) with $V(t)$ replacing $\dot{y}(t)$. Consequently, exponential localization is effected if $V(t)$ is p.e.. As proved in [12], $V(t)$ is p.e. iff $\dot{y}(t)$ is p.e.. Clearly then the convergence proof in [12], is trivial, once as is done in [12], $\dot{y}(t)$ is assumed to be p.e.. There is a potential for p.e. to be lost once the loop is closed to achieve circumnavigation. One key challenge of this paper is to devise a control step that maintains p.e. despite closing the loop. This is done in Section III-B.

B. Control Step

In keeping with our outlined strategy we now propose a control law that forces $y(t)$ to circumnavigate $\hat{x}(t)$, generated through (III4)–(III7). Define

$$\hat{D}(t) = \|y(t) - \hat{x}(t)\| \tag{III9}$$

and the control law

$$\dot{y}(t) = \dot{\hat{x}}(t) - \left[\left(\hat{D}^2(t) - d^2\right)I - A(t)\right](y(t) - \hat{x}(t)) \tag{III10}$$

where $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ obeys four conditions captured in the assumption below (That there exist $A(t)$ satisfying these conditions will be shown later in Section VIII). As will be proved in the next section, this control law forces $\hat{D}(t)$ to converge to d, i.e. the agent takes up the correct distance from the estimated target position, $\hat{x}(t)$. If also $\hat{x}(t)$ converges to $x(t)$, then D converges to $\hat{D}$, hence D converges to d.

Assumption 3.1: (i) There exists a $T > 0$ such that

$$A(t+T) = A(t) \text{ for all } t \tag{III11}$$

and (ii) $A(t)$ is skew symmetric for all t; (iii) $A(t)$ is differentiable everywhere; and (iv) the derivative of the solution of the differential equation

$$\dot{y}^*(t) = A(t)y^*(t) \tag{III12}$$

is p.e. for any arbitrary nonzero value of $y^*(0)$. More precisely, there exist positive $T_1, \alpha_i > 0$ such that for all $t \geq 0$

$$\alpha_1 \|y^*(t)\|^2 I \leq \int_t^{t+T_1} \dot{y}^*(\tau)\dot{y}^*(\tau)^\top d\tau \leq \alpha_2 \|y^*(t)\|^2 I. \tag{III13}$$

We now motivate this algorithm by flagging certain properties that will be derived in the sequel. First, we note that in (III10) the $(\hat{D}^2(t) - d^2)I$ term helps drive $\hat{D}(t)$ to d. The role of the $A(t)$ matrix is to force y to rotate around $\hat{x}$ at a distance d and to induce p.e.. To understand this, note that as $A(t)$ is skew symmetric, for all $\nu \in \mathbb{R}^n$ and $t \geq 0$

$$\nu^\top A(t)\nu = 0. \tag{III14}$$

As shown in the next section, (III14) ensures that the solution of (III12) has the same magnitude at all instants of time. Thus, should $\hat{D}(t)$ converge to d, then $y(t)$ will circumnavigate $\hat{x}(t)$, the localization estimate, at the presecribed distance d. Indeed, Setion IV shows that *regardless of drift in the target*, $\hat{D}(t)$ does indeed converge to d. Further, because of (iv), $\dot{y}(t) - \dot{\hat{x}}(t)$ is asymptotically p.e.. Should $\dot{\hat{x}}(t)$ converge to zero or become ultimately small, then $\dot{y}$ would still be p.e.. Indeed these are the very properties flagged in the introduction. As already noted, a key challenge is to show that $\hat{x}$ does converge to x which in turn requires that $\dot{y}$ be p.e. to begin with.

As shown in Section VIII, in $\mathbb{R}^2$, with E the rotation matrix below, and a any *nonzero* real scalar, $A(t) = aE$ suffices

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{III15}$$

In $\mathbb{R}^3$ however, the selection of $A(t)$ is more complicated, as for (III13) to hold with a constant A, A must be nonsingular. No 3×3 skew symmetric matrix is however nonsingular, thus entailing a periodic $A(t)$, described in Section VIII.

To summarize, the overall system is (II1), (III4)–(III7) and (III10), under Assumption (3.1) and (III9). Further, the z_i, $\hat{x}(t)$, and $y(t)$, serve as the underlying state variables. The stability analysis that follows, is for a general n.

Two final points need to be made. First, as implied in the foregoing, we will show that $\dot{y}(\cdot)$ is in fact p.e.. This is what we mean by "circumnavigation". Combined with (II2) this means that asymptotically, the agent executes a sufficiently rich trajectory at a distance d from the target. Second, while it may appear that the algorithm requires the knowledge of the absolute position y, it in fact allows the agent to select the coordinate frame with respect to which y is chosen. For example the agent may choose $y(0)$, or for that matter $\hat{x}(0)$, to be the origin. Thus, this imposes no more of a burden than that required by algorithms that work with relative as opposed to absolute position information.

IV. Preliminary Analysis

In this section we derive some important relationships and properties that reduce to many of the intermediate properties flagged in the foregoing. *All the results in this section hold regardless of whether the target drifts.*

We first establish a relationship that in the stationary target case, justifies the observation leading to (III8).

Observe from (III4) that $\eta(t) = \dot{z}_1(t)$. Thus, one obtains

$$\dot{\eta}(t) = -\alpha\eta(t) + \dot{y}(t)^\top (y(t) - x(t)) - \dot{x}(t)^\top (y(t) - x(t)) . \tag{IV1}$$

Similarly

$$\dot{m}(t) = -\alpha m(t) + \dot{y}(t)^\top y(t); \tag{IV2}$$
$$\dot{V}(t) = -\alpha V(t) + \dot{y}(t). \tag{IV3}$$
$$\dot{p}(t) = -\alpha p(t) + \dot{x}(t)^\top (V(t) + x(t) - y(t)) \tag{IV4}$$

where

$$p(t) = \eta(t) - m(t) + V^\top(t)x(t). \tag{IV5}$$

The significance of (IV4) under (IV5) is as follows. First when $x(t) \equiv x$, with x a constant, the observation in (III8) follows. More importantly it captures the dependence of the update kernel in (III7) on the target velocity. Recall also that to tackle partial stability issues we will eventually redefine the state space. *The signal* $p(t)$, *will be an important element of the redefined state and* (IV4) *an important part of the state update*.

As foreshadowed earlier, we now present a lemma that shows that the agent located at $y(t)$ moves to a trajectory maintaining a constant distance d from the *estimated* position of the target at position $x(t)$, *even if the target drifts*.

Lemma 4.1: Consider (III10) under Assumption 3.1. Suppose that there exists a constant $\delta > 0$ such that in (III9), $\hat{D}^2(0) > \delta$. Then $\hat{D}^2(t)$ converges exponentially to d^2, and there holds for all $t \geq 0$

$$\hat{D}^2(t) > \min\{\delta, d^2\}. \tag{IV6}$$

Proof: See Appendix A. ∎

Thus we have proved that irrespective of drift, (III10) meets one of its defining objectives: That the agent will ultimately be at a distance d from the estimated location of the target.

Section III-B noted that as $A(t)$ is skew symmetric, the solution of (III12) has constant magnitude. To establish this all we need to show is that the state transition matrix for $A(t)$, i.e. $\Phi(t, t_0)$ that obeys for all t, t_0

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I \tag{IV7}$$

is orthogonal. This is done in Lemma 4.2 below.

Lemma 4.2: Consider $\Phi(t, t_0)$ defined in (IV7). Under Assumption 3.1

$$\Phi^\top(t, t_0)\Phi(t, t_0) = I \quad \text{for all } t, t_0.$$

Proof: See Appendix A. ∎

Another claim made in Section III-B was that (III10) ensures that $\dot{y}(\cdot) - \dot{\hat{x}}(\cdot)$ is p.e. We now demonstrate that fact. For notational convenience define the signal

$$q_1(t) = \dot{y}(t) - \dot{\hat{x}}(t). \tag{IV8}$$

We have the following proposition which again holds regardless of whether the target drifts.

Proposition 4.1: Consider (III10), (II1) and (III9), under Assumption 3.1 and $\hat{D}^2(0) > \delta > 0$. Then the signal $q_1(t)$ defined in (IV8) has the following properties: (i) It is p.e. (ii) It and its derivative are bounded.

Proof: See Appendix A. ∎

We end this section by commenting on the significance of $q_1(t)$ to the development in the sequel. Recall the need to redefine the state space to invoke standard inverse Lyapunov theorems so that we can deal with drift. One reason why these theorems do not apply is that even for stationary targets, a part of the state space converges to a trajectory that is not completely specified, as all we can say about the asymptotic trajectory of $y(\cdot)$ is that it obeys $\|y(\cdot) - \hat{x}(\cdot)\| = d$. This alone is not enough to invoke standard inverse Lyapunov theorems requiring more specificity. In the kernel of the redefined state space we will use to circumvent this difficulty, $q_1(t)$ will generate a time varying parameter. Even though technically $q_1(t)$ is a function of the state variables of the original state vector, its properties required to prove stability of this redefined state space are those claimed in Proposition 4.1.

V. Reduced State Space

To this point we have shown that a part of the objectives of our two-pronged strategy, namely that $\hat{D}(t)$ converges exponentially to d, is met regardless of whether $\dot{x}$ is zero. We have also shown, again without regard to the presence of drift, that the signal $q_1(t) = \dot{y}(t) - \dot{\hat{x}}(t)$ is p.e. Thus, to show that our remaining objectives are met, all we need to show is that $\hat{x}(t)$ converges exponentially to x in the drift free case, and to within an estimation error proportional to the maximum drift speed, provided this speed is small enough. This will in turn show that $D(t)$ converges exponentially to d, in the drift free case and is close to d, under slow enough drift.

Taken at face value, a natural state space describing our algorithm is that involving the state vector: $[z_1, z_2, z_3^\top, \tilde{x}^\top, y^\top]^\top$, where $\tilde{x}$ is the localization error

$$\tilde{x}(t) = \hat{x}(t) - x(t). \tag{V1}$$

It is however, difficult to tackle drift with this choice of state vector. Specifically, by design, even in the drift free case only $\tilde{x}$ and z_1 converge exponentially to a point. The remaining part of the state vector comprising, z_2, z_3 and y are not designed to converge to a point, but rather to a trajectory that is only partially defined. This makes it difficult to directly leverage conventional inverse Lyapunov arguments.

To address this problem we recognize that all that is at issue is the behavior of $\tilde{x}$, while $y(t)$, which determines z_2 and z_3, is pertinent only to the extent that it indirectly facilitates the convergence of $\tilde{x}$. In fact as will be evident in the sequel the convergence of $\tilde{x}$ hinges on $V(\cdot)$ in (III6) being p.e.. This in turn is ensured by $\dot{y}$ being p.e., *regardless of drift*. The key technical difficulty lies in ensuring this latter fact particularly when the target drifts.

To sidestep this difficulty in Section V-A we present a reduced state vector whose elements exponentially converge to zero in the drift free case. The remaining state variables appear in the system equations governing the behavior this reduced state vector as *time varying parameters*. Section V-B derives certain properties that these parameters satisfy without having to first demonstrate convergence. These properties suffice to prove stability.

A. State Vector Redefined

The reduced state vector is

$$\xi(t) = \left[w(t)^\top \tilde{x}(t)^\top p(t)\right]^\top \tag{V2}$$

where $p(t)$ is as in (IV5), and for arbitrary $w(0)$, $w : \mathbb{R} \to \mathbb{R}^n$ obeys

$$\dot{w}(t) = -\alpha w(t) + \dot{\tilde{x}}(t). \tag{V3}$$

It turns out that all our objectives are met if in the drift free case ξ converges exponentially, to zero. Informally, $\tilde{x}(t)$ going to zero implies that $\hat{x}(t)$ goes to x, i.e. because of Lemma 4.1, $D(t)$ goes to d. Thus all signals are bounded. Furthermore, because of (III7) and (III8), $\dot{\hat{x}}(t)$ goes to zero. Then because of Proposition 4.1, $\dot{y}$ is p.e.

Consequently, a clear virtue of this reduced state space is that we can invoke standard inverse Lypunov theory to tackle the case with drift. Instead of being NLPTV the system of equations governing ξ is *nonlinear time varying (NLTV)* with two time varying parameters: $q_2 : \mathbb{R} \to \mathbb{R}^n$, and $q : \mathbb{R} \to \mathbb{R}^n$ obeying

$$\dot{q}_2(t) = -\alpha q_2(t) + q_1(t), \quad q_2(0) = V(0) - w(0) \tag{V4}$$

$$\dot{q}(t) = -\alpha q(t) + \dot{x}(t); \quad q(0) = 0. \tag{V5}$$

As noted in the previous section, $q_1(t)$ helps generate a crucial parameter, namely q_2. Though q_2 and q are signals related to the state vectors, it turns out *that all that is important about them is that $q_2 + q$ be p.e.*. Observe also that the signals z_i and y do not directly appear in ξ. Rather the features of y critical to establishing stability properties, are captured by the time varying parameter q_2.

To establish the differential equations that govern the reduced state space, we first expose a simple relationship between, V, w, q_2, and q.

Lemma 5.1: Consider (III6), (IV8), (V1), and (V3)–(V5). Then for all t, there holds

$$V(t) = w(t) + q_2(t) + q(t). \tag{V6}$$

Proof: From (IV8) (V1), and (V3)–(V5) one obtains

$$\begin{aligned} \dot{w} + \dot{q}_2 + \dot{q} &= -\alpha w + \dot{\tilde{x}} - \alpha q_2 + q_1 - \alpha q + \dot{x} \\ &= -\alpha(w + q_2 + q) + \dot{\tilde{x}} + \dot{y} - \dot{\tilde{x}} \\ &= -\alpha(w + q_2 + q) + \dot{y} \end{aligned}.$$

Then the result follows from (IV3) and the fact that $w(0) + q_2(0) + q(0) = V(0)$. ■

To complete the derivation of the new state variable equations, observe from (III7), (IV5) and (V1) that

$$\begin{aligned} \dot{\tilde{x}}(t) &= -\gamma V(t)\left(\eta(t) - m(t) + V^\top(t)\hat{x}(t)\right) - \dot{x}(t) \\ &= -\gamma V(t)\left(\eta(t) - m(t) + V^\top(t)x(t)\right) \\ &\quad - \gamma V(t)V^\top(t)\tilde{x}(t) - \dot{x}(t) \\ &= -\gamma V(t)p(t) - \gamma V(t)V^\top(t)\tilde{x}(t) - \dot{x}(t). \end{aligned}$$

Thus, using (IV4), (V3) and Lemma 5.1 we have that

$$\begin{aligned} \dot{w}(t) &= -\alpha w(t) - \gamma\left(w(t) + q_2(t) + q(t)\right) \\ &\quad \times \left(w(t) + q_2(t) + q(t)\right)^\top \tilde{x} \\ &\quad - \gamma\left(w(t) + q_2(t) + q(t)\right)p(t) - \dot{x}(t) \\ \dot{\tilde{x}}(t) &= -\gamma\left(w(t) + q_2(t) + q(t)\right) \\ &\quad \times \left(w(t) + q_2(t) + q(t)\right)^\top \tilde{x}(t) \\ &\quad - \gamma\left(w(t) + q_2(t) + q(t)\right)p(t) - \dot{x}(t) \\ \dot{p}(t) &= \dot{x}(t)^\top\left[\hat{x}(t) - \tilde{x}(t) - y(t) + w(t) + q_2(t) + q(t)\right] \\ &\quad - \alpha p(t). \end{aligned} \tag{V7}$$

These are the governing equations for the new state space when the *target drifts*; $q(t)$ and $q_2(t)$ appear as time varying parameters. They capture the part of the state space that does not go to zero.

We next specialize this system to the *driftless case*. Note that, as $q(0) = 0$, for $\dot{x}(t)$ identically zero, one has from Lemma 5.1 and (V5) that

$$V(t) = w(t) + q_2(t). \tag{V8}$$

Thus in the driftless case of $\dot{x}(t) = 0$ (V7) becomes

$$\begin{aligned} \dot{w}(t) &= -\alpha w(t) - \gamma\left(w(t) + q_2(t)\right)\left(w(t) + q_2(t)\right)^\top \tilde{x} \\ &\quad - \gamma\left(w(t) + q_2(t)\right)p(t) \\ \dot{\tilde{x}}(t) &= -\gamma\left(w(t) + q_2(t)\right)\left(w(t) + q_2(t)\right)^\top \tilde{x}(t) \\ &\quad - \gamma\left(w(t) + q_2(t)\right)p(t) \\ \dot{p}(t) &= -\alpha p(t). \end{aligned} \tag{V9}$$

Even in the drift free case of (V9), the resulting reduced order system is NLTV rather than NLPTV, and $q_2(t)$ now appears as a time varying parameter.

Our goal will be to show that the system (V9) is exponentially stable. Then from Lemma 4.1 one can immediately draw the conclusion that $y(t)$ goes to a distance d from $x(t)$ exponentially fast and moves around $x(t)$ with radius d. To tackle drift we will then invoke inverse Lyapunov theorems. To this end we will treat (V7) as a perturbed version of (V9). Observe the perturbation comes directly through an affine perturbation manifested through the $\dot{x}(t)$ terms. It also comes indirectly through the parameter $q(t)$ that exponentially vanishes when $\dot{x}(t)$ is zero.

B. Key Properties of Time Varying Parameters

Recall, our earlier assertion that $q_2(t)$ and $q(t)$ model the effect of the nonvanishing parts of the original state space. As is shown in subsequent sections (V9) is exponentially stable if $q_2(t)$ is p.e.. Further the stability of (V7) requires that $q_2(t) +$

$q(t)$ is p.e.. In this section we demonstrate these properties. First a proposition demonstrating the p.e. of q_2.

Proposition 5.1: Suppose $\alpha > 0$, and the conditions of Proposition 4.1 hold. Then the signal $q_2(t)$ defined in (V4) has the following properties: (i) It is p.e. (ii) It and its derivative are bounded.

Proof: See Appendix B. ■

We next show that under sufficiently slow drift, $q_2 + q$ is p.e. as well.

Proposition 5.2: Suppose $\alpha > 0$, the conditions of Proposition 4.1 hold, for some ε and all t, $\|\dot{x}(t)\| \leq \varepsilon$, and q_2 and q are as in (V4) and (V5). Define $\bar{q}(t) = q_2(t) + q(t)$. Then there exists a $\bar{\varepsilon}$, such that for all $\varepsilon \leq \bar{\varepsilon}$, $\bar{q}(t)$ is p.e. and bounded.

Proof: See Appendix B. ■

Remark 5.1: From the proof of this proposition in Appendix B, it is evident that, $\beta_3 - \bar{\varepsilon}^2 T_2/\alpha^2$ and $\beta_4 + \bar{\varepsilon}^2 T_2/\alpha^2$ (see (B.1) and (B.2)) represent a lower bound on β_1 and an upper bound on β_2 respectively. Both are independent of ε. The integrals representing the p.e. conditions of q_2 and $q_2 + q$ are over identical intervals.

Qualitatively, the value of ε that ensures the p.e. of $q_2 + q$ is one but, not the only, condition that defines how fast a drift is permissible.

VI. Stability for Stationary Target

In this section we establish the fact that our algorithm achieves the circumnavigation objective in the driftless case by first establishing the exponential stability of (V9). As is evident from the statement of the proposition below, the key requirement for convergence is that $q_2(t)$ be p.e., a fact already established in the previous section.

Proposition 6.1: The system (V9) with the state variables $w : \mathbb{R} \to \mathbb{R}^n$, $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ and $p : \mathbb{R} \to \mathbb{R}$, and $\alpha > 0$ and $\gamma > 0$, is globally exponentially asymptotically stable if $q_2 : \mathbb{R} \to \mathbb{R}^n$ is bounded and p.e..

Proof: See Appendix C. ■

Remark 6.1: Since the system is exponentially asymptotically stable, there exist $\lambda_i > 0$ such that for all t_0 and $t \geq t_0$

$$\|\xi(t)\| \leq \lambda_1 \|\xi(0)\| e^{-\lambda_2(t-t_0)}.$$

Faster convergence results from a smaller λ_1 and larger λ_2. These depend exclusively on the p.e. parameters of q_2, α and γ. In particular upper bounds on λ_1 and lower bounds on λ_2 can be found that connote a convergence rate that increases with β_1 (see (B.1)) and α, and declines with β_2 and T_2. The dependence on γ is more complicated.

We can now state the main result of this section.

Theorem 6.1: Consider the system described by the equations (II1), and (III4)–(III7), (III9), (III10) with assumption 3.1 in force, $\dot{x}(t) = 0$, for all t, and for some $\delta > 0$, $\hat{D}(t) > \delta$. Then $D(t)$ converges to d exponentially.

Proof: Evidently, (V9) holds. Because of Proposition 5.1 q_2 is p.e.. Thus, from Proposition 6.1, and (V1), $\hat{x}$ converges exponetially to x, and Lemma 4.1, proves the result. ■

This proves the exponential stability of our algorithm in the driftless case, with the minor caveat that $\hat{x}(0) \neq y(0)$. Because of Lemma 4.1, $y(t)$ does not converge to a point but rather moves in an orbit around the traget.

VII. Stability Under Slow Drift

So far we have established exponential stability for the case where $\dot{x} = 0$. Now we consider a varying $x(t)$, subject to the following assumption:

Assumption 7.1: The target trajectory is differentiable and there exists $\varepsilon \in [0, \infty)$ such that for all $t \in \mathbb{R}$

$$\|\dot{x}(t)\| \leq \varepsilon. \tag{VII1}$$

We will first prove the stability of (V7) and then tie it back to the circumnavigation problem. We rewrite the system defined by (V7) as

$$\dot{\xi}(t) = F(\xi, t) + g_1(\xi, t) + g_2(\xi, t) \tag{VII2}$$

where

$$g_1(\xi, t) = \begin{bmatrix} 0 & 0 & \dot{x}(t)^\top [-\tilde{x}(t) + w(t)] \end{bmatrix} \tag{VII3}$$

$$\begin{aligned} g_2(\xi, t) = [&-\dot{x}(t) \quad -\dot{x}(t) \\ &\dot{x}(t)^\top [\hat{x}(t) - y + q_2(t) + q(t)]] \end{aligned} \tag{VII4}$$

$$\begin{aligned} F(\xi, t) &= [F_1(\xi, t) F_2(\xi, t) F_3(\xi, t)]^\top, \\ F_1(\xi, t) &= -\alpha w(t) \\ &\quad - \gamma (w(t) + q_2(t) + q(t)) \\ &\quad \times (w(t) + q_2(t) + q(t))^\top \tilde{x} \\ &\quad - \gamma (w(t) + q_2(t) + q(t)) p(t), \\ F_2(\xi, t) &= -\gamma (w(t) + q_2(t) + q(t)) \\ &\quad \times (w(t) + q_2(t) + q(t))^\top \tilde{x}(t) \\ &\quad - \gamma (w(t) + q_2(t) + q(t)) p(t), \\ F_3(\xi, t) &= -\alpha p(t). \end{aligned} \tag{VII5}$$

Observe $F(\xi, t)$ equals the kernel of (V9), with $q_2(t)$ replaced by $q_2(t) + q(t)$. As the previous section has established that with sufficiently small $\dot{x}(t)$, $q_2(t) + q(t)$ is p.e., by invoking Proposition 6.1, one can conclude that had g_1 and g_2 been zero, (VII2) would have been exponentially stable. The g_i act as perturbations that will be treated by invoking standard inverse Lyapunov theorems.

Proposition 7.1: Consider the system defined by (VII2) with the state variables $w : \mathbb{R} \to \mathbb{R}^n$, $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ and $p : \mathbb{R} \to \mathbb{R}$, a time varying parameter $q : \mathbb{R} \to \mathbb{R}^n$ as in (V5), assumption 7.1 in force, and $\alpha > 0$ and $\gamma > 0$. Suppose $q_2 : \mathbb{R} \to \mathbb{R}^n$ is bounded and p.e. and there is a constant K_3 such that $\|y(t) - \hat{x}(t)\| \leq K_3$ for all t. Then there exist positive constants $\bar{\varepsilon}$ and K such that for all $\varepsilon < \bar{\varepsilon}$ $\limsup_{t \to \infty} \|\xi\| \leq K\varepsilon$. Further K is independent of ε, and convergence is uniform in the initial time.

Proof: See Appendix C. ■

Now we present the main result of this section.

Theorem 7.1: Consider the system described by the equations (II1), and (III4)–(III7), (III9), (III10) with assumptions 3.1 and 7.1 in force, and for some $\delta > 0$, $\hat{D}(t) > \delta$. Then there exist positive constants $\bar{\varepsilon}$ and K such that for all $\varepsilon < \bar{\varepsilon}$

$$\limsup_{t \to \infty} \|D(t) - d\| \leq K\varepsilon.$$

Proof: Note (V7) holds. From Proposition 5.1, q_2 is p.e.. Thus, from Proposition 7.1, and (V1), for sufficiently small $\bar{\varepsilon}$, there exists K independent of ε, such that $\hat{x}$ is ultimately bounded by $K\varepsilon$ and Lemma 4.1, proves the result. ■

Thus all the circumnavigation objectives are met. Qualitatively, $\bar{\varepsilon}$, the upper bound on the target velocity, depends on d, and α_i and T_1 in (III13). These ultimately determine the p.e. parameters of q_2. Based on these $\bar{\varepsilon}$ must be small enough to ensure that, (a) $q_2 + q$ is p.e.; (b) β in the proof of Proposition 7.1 is positive; and $K_5\bar{\varepsilon}$, in the proof of Theorem 7.1 is small enough. Under these conditions convergence is global modulo the requrement that $y(0) \neq \hat{x}(0)$.

VIII. CHOOSING $A(t)$

In this section we focus on the selection of $A(t)$ to satisfy Assumption 3.1. Consider first $n = 2$, we show that with E as in (III15) the matrix $A(t) = aE$ obeys the requirements of Assumption 3.1. Indeed consider the Lemma below.

Lemma 8.1: With E as in (III15), and a nonzero $a \in \mathbb{R}$ consider $\zeta : \mathbb{R} \to \mathbb{R}^2$ obeying

$$\dot{\zeta}(t) = aE\zeta(t). \tag{VIII1}$$

Denote $\zeta = [\zeta_1, \zeta_2]^\top$. Define $\beta(t_0)$ as the argument of the complex number $\zeta_1 + \boldsymbol{j}\zeta_2$. Then there holds for all $t \geq t_0 \geq 0$

$$\begin{aligned}\zeta(t) = \|\zeta(t_0)\| \left[\cos\left(a(t-t_0)+\beta(t_0)\right),\right.\\ \left.\sin\left(a(t-t_0)+\beta(t_0)\right)\right]^\top.\end{aligned} \tag{VIII2}$$

Proof: Follows from the facts that $\zeta(t_0) = \|\zeta(t_0)\|[\cos(\beta(t_0)), \sin(\beta(t_0))]^\top$, and that the state transition matrix corresponding to (VIII1) is:

$$e^{aEt} = \begin{bmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{bmatrix}.$$

■

The fact that (VIII2) satisfies (III13) with y^* identified with ζ, is trivial to check. It is also clear that under this selection, $y(t)$ circumnavigates x with an angular speed of $|a|$.

To address the $n = 3$ case we first preclude the possibility that $A(t)$ can be a constant matrix. Indeed observe that no real skew-symmetric matrix in $\mathbb{R}^{3\times 3}$ can be nonsingular, as if λ is an eigenvalue of a skew symmetric matrix then so is $-\lambda$. Thus for any odd n, an $n \times n$ skew symmetric matrix must have a zero eigenvalue. To complete the argument we present the following Lemma.

Lemma 8.2: Suppose in (III12) $A(t) \equiv A$ for all t and A is singular. Then (III13) cannot hold.

Proof: If A is singular, then e^{At} has an eigenvalue at one. Thus there exists a $y^*(0)$ such that for all t, $y^*(t) = e^{At}y^*(0)$ is a constant, i.e. for this $y^*(0)$, $\dot{y}^* \equiv 0$. ■

Thus, we must search for a periodic $A(t)$ to meet the requirements of Assumption 3.1. Effectively, the $A(t)$ we will choose will switch periodically between the two 3×3 matrices

$$B = \begin{bmatrix} 0 & 0 \\ 0 & bE \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 0 \\ 0 & cE \end{bmatrix} \tag{VIII3}$$

b and c being real nonzero scalars. *Observe, B rotates y on the plane defined by $y_3 = 0$. Likewise C rotates y on the plane defined by $y_1 = 0$. This switching can be shown to achieve the required condition, and its effect is illustrated in the next section through simulations*. However, to ensure that the resulting matrix is differentiable, we require a differentiable transition between B and C. To achieve this define a nondecreasing $g : \mathbb{R} \to \mathbb{R}$, that obeys

$$g(t) = 0 \quad \forall t \leq 0 \tag{VIII4}$$
$$g(t) = 1, \quad \forall t \geq 1, \text{ and} \tag{VIII5}$$
$$g(t) \text{ is differentiable } \forall t. \tag{VIII6}$$

An example of such a $g(t)$ is

$$g(t) = \begin{cases} \frac{1}{2}\left(1-\cos(\pi t)\right) & 0 \leq t \leq 1 \\ 0 & t < 0 \\ 1 & t > 1. \end{cases} \tag{VIII7}$$

Clearly this satisfies (VIII4) and (VIII5). Further (VIII6) holds as

$$\lim_{t\to 0_+} \dot{g}(t) = \lim_{t\to 1_-} \dot{g}(t) = 0.$$

Now, for nonzero scalars b and c, we will select $A(t)$ as follows. For a suitably small $\rho > 0$, define

$$\bar{T}_1 = \rho, \quad \bar{T}_2 = \rho + \frac{\pi}{|b|}, \quad \bar{T}_3 = 2\rho + \frac{\pi}{|b|} \tag{VIII8}$$

and

$$\bar{T}_4 = 3\rho + \frac{\pi}{|b|},\ \bar{T}_5 = 3\rho + \frac{\pi}{|b|} + \frac{\pi}{|c|},\ T = \bar{T}_6 = 4\rho + \frac{\pi}{|b|} + \frac{\pi}{|c|}. \tag{VIII9}$$

For all t, let $K_T(t)$ denote the largest integer k satisfying $t \geq kT$ and let $r_T(t) = t - K_T(t)T$. Then define $A(t)$ as

$$A(t) = \begin{cases} g\left(\frac{t}{\rho}\right)B & 0 \leq r_T(t) \leq \bar{T}_1 \\ B & \bar{T}_1 \leq r_T(t) \leq \bar{T}_2 \\ \left(1 - g\left(\frac{t-\bar{T}_2}{\rho}\right)\right)B & \bar{T}_2 \leq r_T(t) \leq \bar{T}_3 \\ g\left(\frac{t-\bar{T}_3}{rho}\right)C & \bar{T}_3 \leq r_T(t) \leq \bar{T}_4 \\ C & \bar{T}_4 \leq r_T(t) \leq \bar{T}_5 \\ \left(1 - g\left(\frac{t-\bar{T}_5}{\rho}\right)\right)C & \bar{T}_5 \leq r_T(t) \leq \bar{T}_6 = T. \end{cases} \tag{VIII10}$$

Observe that (VIII10) automatically satisfies (i–iii) of Assumption 3.1. That it satisfies (iv) as well, is now proved.

Theorem 8.1: Consider (III12) with $A(t)$ defined in (VIII8)–(VIII10). Then for every pair of nonzero b, c there exists a ρ^* such that (III13) holds for all $0 < \rho \leq \rho^*$.

Proof: See Appendix D. ■

IX. SIMULATIONS

In this section, we present simulation studies of the behavior of the circumnavigation system (III4)–(III10). We consider three scenarios in $\mathbb{R}^2$ and one in $\mathbb{R}^3$.

In the first simulation, depicted in Fig. 1, we study the case where $x = [0.5, 3]^\top$, $d = 2$, and $y(0) = [8, 5]^\top$. A closer

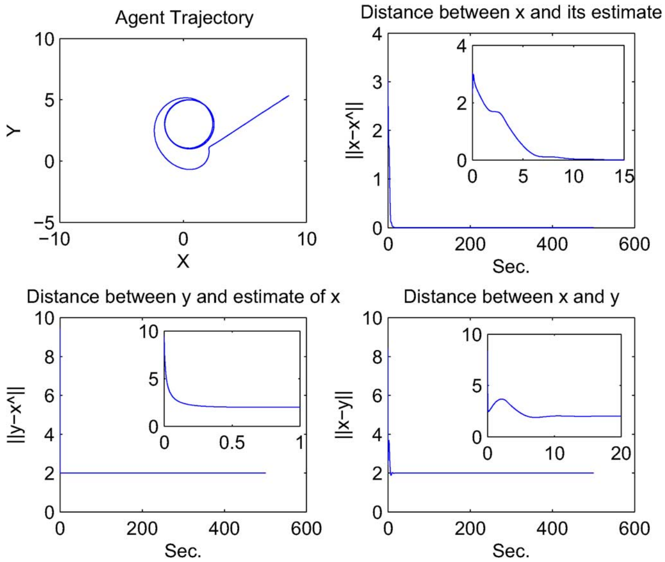

Fig. 1. Agent trajectory in $X-Y$ plane. $\|y(t)-\hat{x}(t)\|$, $\|y(t)-x(t)\|$, and $\|x(t)-\hat{x}(t)\|$ for the case where the target is stationary and there is not any noise present in the distance measurements.

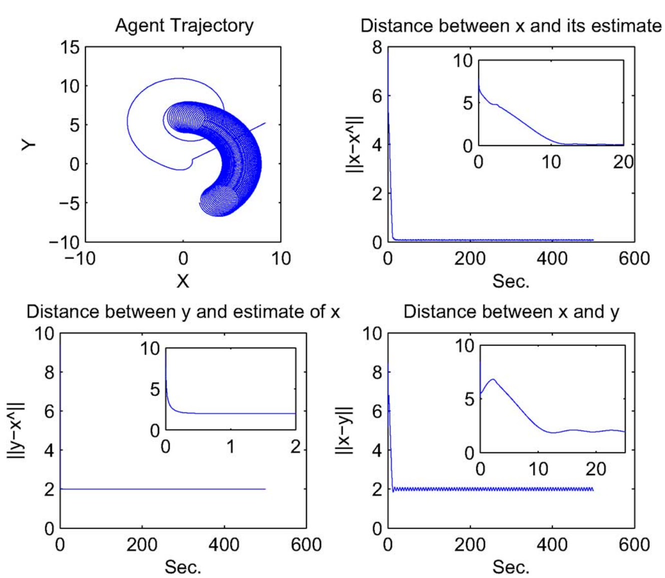

Fig. 2. Agent trajectory in $X-Y$ plane. $\|y(t)-\hat{x}(t)\|$, $\|y(t)-x(t)\|$, and $\|x(t)-\hat{x}(t)\|$, for the case where the target is undergoing on a drifting motion on a circle ($\dot{x}(t)\neq 0$), and the distance measurements are noise free.

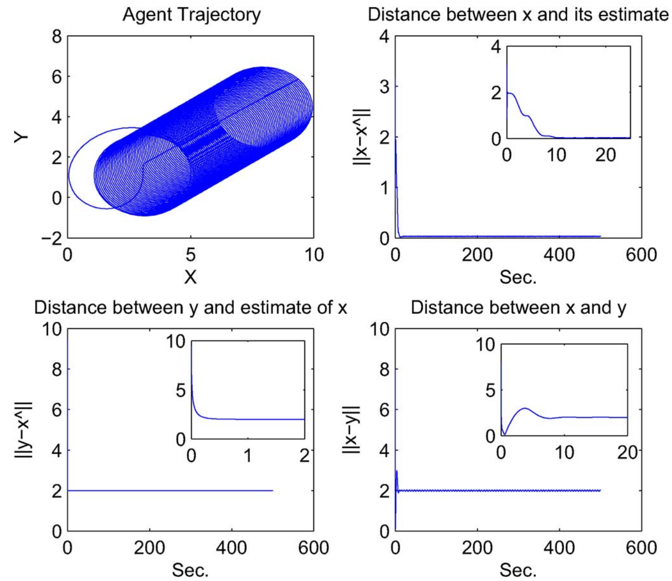

Fig. 3. Agent trajectory in $X-Y$ plane. $\|y(t)-\hat{x}(t)\|$, $\|y(t)-x(t)\|$, and $\|x(t)-\hat{x}(t)\|$ for the case where the target is drifting on a line with constant velocity, and the distance measurements are noise free.

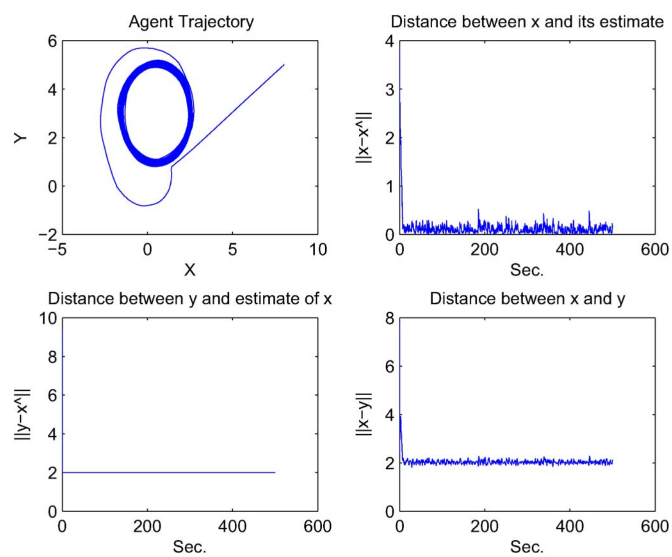

Fig. 4. Agent trajectory in $X-Y$ plane. $\|y(t)-\hat{x}(t)\|$, $\|y(t)-x(t)\|$, and $\|x(t)-\hat{x}(t)\|$ for the case where the target is stationary but the distance measurements are corrupted by noise.

look at the agent trajectory reveals a very small radius turn near the point $[2,1]^\top$. The reason for this behavior is the following. The term $(\hat{D}^2(t)-d^2)(y(t)-\hat{x}(t))$ in (III10) is designed to force $y(t)$ to move on a straight line trajectory in a manner that drives $\hat{D}$ to d. The second term $A(t)(y(t)-\hat{x}(t))$ forces $y(t)$ to rotate around $\hat{x}(t)$. Initially the first term is dominant, and the agent quickly travels a long distance on an almost straight line. By the time the agent reaches $[2,1]^T$, the rotational motion component becomes comparable to the straight line motion component; hence the effect of this change shows itself as a sharp turn.

In the second simulation, shown in Fig. 2, we study the behavior of the system when the target slowly drifts on a circle centered at the origin with angular velocity equal to 0.005 rad/sec. The agent maintains its distance from the target in a neighborhood of the desired distance. Notice that the speed of the target is always much less than the speed of the agent.

The third simulation, Fig. 3, depicts the algorithm coping with a target moving with a constant velocity. Again the agent maintains its distance from the target in a neighborhood of the desired distance.

The fourth, Fig. 4, considers the case where the target is stationary and the distance measurement is noisy: it is assumed that $\ln\bar{D}(t)=\ln D(t)+\mu(t)$, where $\bar{D}(t)$ is measurement of the distance $D(t)$ and μ is a strict-sense stationary random process with $\mu(t)\sim N(0,\sigma^2), \forall t$. Evidently, the control law is still successful in moving the agent to an orbit with the distance to the source kept close to its desired value.

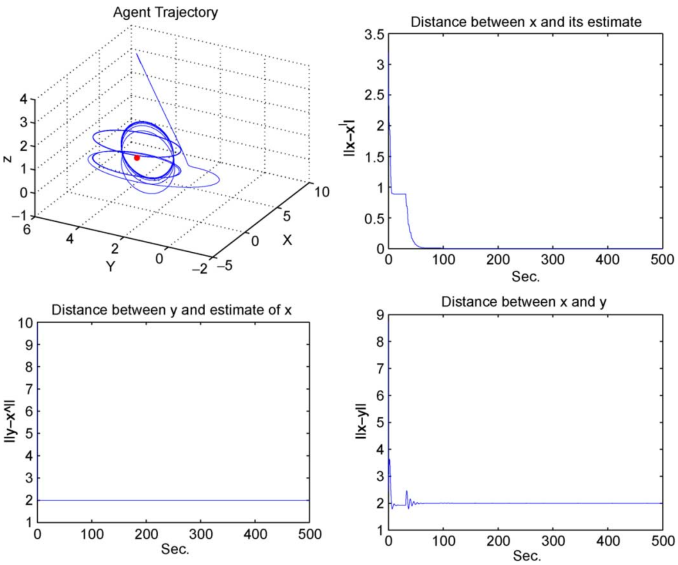

Fig. 5. Agent trajectory in $X-Y-Z$ plane. $\|y(t)-\hat{x}(t)\|$, $\|y(t)-x(t)\|$, and $\|x(t)-\hat{x}(t)\|$ for the case where the target is stationary and there is no noise present in the distance measurements. Moreover, the agent is forced to have constant speed.

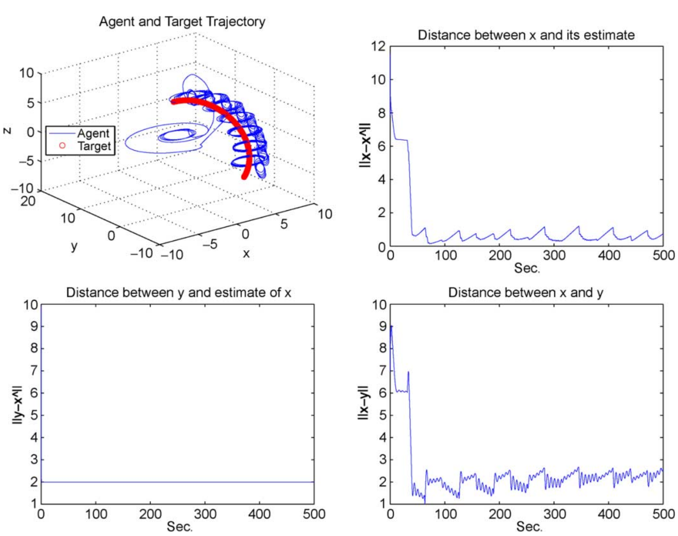

Fig. 6. Agent trajectory in $X-Y-Z$ plane. $\|y(t)-\hat{x}(t)\|$, $\|y(t)-x(t)\|$, and $\|x(t)-\hat{x}(t)\|$ for the case where the target is undergoing a drifting motion, and the distance measurements are noise free.

Fig. 5 depicts the case where $x=[0.5,3,1]^\top$, $d=2$, and in (VIII3), $b=c=1$. Some features of the agent trajectory are noteworthy. First in the transient phase three distinct phenomena are observed. As in the 2-D case, while $\hat{D}(t)-d(t)$ is large the agent heads toward the target pretty much in a straight line. Once $\hat{D}(t)-d(t)$ becomes small the rotational effect of $A(t)$ in (III10) dominates. Note in the design of $A(t)$ in the three dimensional case the agent alternately rotates parallel to the X-Y and the Y-Z planes. The transient phase concludes after just one such pair of rotations. Subsequently the agent circumnavigates the target by alternatingly rotating along the X-Y and the Y-Z planes.

Finally, Fig. 6 depicts a 3-D example with a slowly drifting target with trajectory, $x = 6[\sin(0.005t),\cos(0.005t),\cos(0.005t)]^T$ and with initial agent position: $y(0)=[8,5,3]^T$. The desired distance is $d=2$, and in (VIII3), $b=c=1$. The transient phase has the same features as the stationary case above. At steady state, as the agent pursues the drifting target, alternating motion parallel to the X-Y and the Y-Z plains persists.

X. Conclusion

We have proposed an algorithm for circumnavigating a target at an unknown position by a single agent, at a pre-defined distance from the target, using only the measurements of the agent's distance to the target. Stability has been established when the target is stationary and when it is undergoing a slow drift. Furthermore, in simulations the performance of the method in the presence of noise and in the situations where the source is undergoing a drifting motion is presented. A possible extension of the current scheme is to consider the cases where more than one agent is present.

Appendix

A. *Proofs of Results in Section IV*

Proof of Lemma 4.1: Because of (III14), (III10) implies

$$\begin{aligned}\frac{d}{dt}\left\{\hat{D}^2(t)-d^2\right\} &= 2\left(\dot{y}(t)-\dot{\hat{x}}(t)\right)^\top(y(t)-\hat{x}(t))\\ &= -2\left(\hat{D}^2(t)-d^2\right)\hat{D}^2(t). \qquad \text{(A1)}\end{aligned}$$

Observe that $\hat{D}$ is bounded and continuous. Consider first the case where $\delta > d^2$. Then the derivative above is initially negative, i.e. $\hat{D}^2(t)$ declines in value. By its continuity, for $\hat{D}^2(t)$ to become less d^2, at some point it must equal d^2, when $\hat{D}(t)$ will stop changing. Since throughout this time $\hat{D}^2(t)\geq d^2$, convergence of $\hat{D}^2(t)$ to d^2 occurs at an exponential rate and $\hat{D}^2(t)\geq d^2$ for all t. If $\delta\leq d^2$, then $\hat{D}^2(t)>\delta$ for all t, as the derivative of $\hat{D}^2(t)$ is nonnegative. Again exponential convergence of $\hat{D}^2(t)$ to d^2 occurs.

Proof of Lemma 4.2: Consider (III12). Then for all t,t_0, and $y^*(t_0)\in\mathbb{R}^n$, $y^*(t)=\Phi(t,t_0)y^*(t_0)$. Further because of (III14)

$$\frac{d}{dt}\left\{y^{*\top}(t)y^*(t)\right\}=2y^{*\top}(t)A(t)y^*(t)=0.$$

Thus, the result holds as for all t,t_0, and all $y^*(t_0)$, $\|y^*(t)\|=\|y^*(t_0)\|$.

Proof of Proposition 4.1: To prove (i) we have to show that there exist $\bar{\alpha}_1,\bar{\alpha}_2,T_2>0$, such that for all $t\geq 0$ there holds

$$\bar{\alpha}_1 I\leq\int_t^{t+T_2} q_1(\tau)q_1^\top(\tau)d\tau\leq\bar{\alpha}_2 I. \qquad \text{(A2)}$$

The upper bound will follow if we prove (ii). Thus we first focus on proving the lower bound. A consequence of Assumption 3.1 is that for all $\phi\in\mathbb{R}^n$, and any unit $\theta\in\mathbb{R}^n$

$$\alpha_1\|\phi\|^2\leq\int_t^{t+T_1}\left|\theta^\top A(\tau)\Phi(\tau,t)\phi\right|^2 d\tau. \qquad \text{(A3)}$$

Further because of (III10) for all $t_1 \geq 0$ and $t \geq t_1$

$$\dot{y}(t) - \dot{\hat{x}}(t) = A(t)\Phi(t,t_1)\left(y(t_1) - \hat{x}(t_1)\right) - A(t)\int_{t_1}^{t} \Phi(t,\tau)\left(\hat{D}^2(\tau) - d^2\right)\left(y(\tau) - \hat{x}(\tau)\right)d\tau. \quad \text{(A4)}$$

Assumption 3.1 ensures that $A(t)$ is bounded. Thus there exists M_2 such that

$$\|A(t)\| \leq M_2 \quad \forall t. \quad \text{(A5)}$$

Further because of Lemma 4.1, there is a $\bar{c} > 0$, such that for every $\bar{\epsilon} > 0$, there is a t_2 such that for all $t \geq t_2$

$$|d - \|y(t) - \hat{x}(t)\|| \leq \bar{\epsilon} \quad \text{(A6)}$$

$$|\hat{D}^2(t) - d^2| \leq \bar{\epsilon}e^{-\bar{c}(t-t_2)}. \quad \text{(A7)}$$

Thus because of Lemma 4.2, and (A4)–(A7) for every unit $\theta \in \mathbb{R}^n$ and $t \geq t_2$

$$\left|\theta^\top\left(\dot{y}(t) - \dot{\hat{x}}(t)\right)\right| \geq \left|\theta^\top A(t)\Phi(t,t_2)\left(y(t_2) - \hat{x}(t_2)\right)\right| - \frac{\bar{\epsilon}(d+\bar{\epsilon})M_2}{\bar{c}}.$$

Thus, because of Lemma 4.2, (A5) and (A6), there exist $\bar{K}_i$ all positive such that for all $t \geq t_2$ there holds

$$\left|\theta^\top\left(\dot{y}(t) - \dot{\hat{x}}(t)\right)\right|^2 \geq \left|\theta^\top A(t)\Phi(t,t_2)\left(y(t_2) - \hat{x}(t_2)\right)\right|^2 - \sum_{i=1}^{4}\bar{K}_i\bar{\epsilon}^i. \quad \text{(A8)}$$

Choose $T_2 = T_1 + t_2$. Then because of (A3) and (A8) for all $t > 0$, there holds

$$\begin{aligned}
&\int_t^{t+T_2}\left|\theta^\top\left(\dot{y}(\tau) - \dot{\hat{x}}(\tau)\right)\right|^2 d\tau \\
&\geq \int_{t+t_2}^{t+T_2}\left|\theta^\top\left(\dot{y}(\tau) - \dot{\hat{x}}(\tau)\right)\right|^2 d\tau \\
&\geq \int_{t+t_2}^{t+T_2}\left|\theta^\top A(\tau)\Phi(\tau,t_2)\left(y(t_2) - \hat{x}(t_2)\right)\right|^2 d\tau \\
&\quad - T_1\sum_{i=1}^{4}\bar{K}_i\bar{\epsilon}^i \\
&\geq \alpha_1(d-\bar{\epsilon})^2 - T_1\sum_{i=1}^{4}\bar{K}_i\bar{\epsilon}^i \\
&\geq \alpha_1 d^2 - 2\alpha_1 d\bar{\epsilon} - T_1\sum_{i=1}^{4}\bar{K}_i\bar{\epsilon}^i.
\end{aligned}$$

Then the left inequality in (A2) follows by choosing $\bar{\epsilon}$ so that $T_1\sum_{i=1}^4 \bar{K}_i\bar{\epsilon}^i + 2\alpha_1 d\bar{\epsilon} \leq \alpha_1 d^2/2$.

The boundedness of q_1 and hence the upper bound of the hypothesized p.e. condition in (i), follows from (III10), boundedness of $A(t)$, Lemma 4.1 and Lemma 4.2. The proof of the boundedness of the derivative of $q_1(\cdot)$ follows from the differentiability of $A(t)$, Lemma 4.1, (A1) and by noting from (III10) that:

$$\dot{q}_1(t) = -\left[\left(\hat{D}^2(t) - d^2\right)I - A(t)\right]q_1(t) - \left[2\hat{D}(t)\dot{\hat{D}}(t)I - \dot{A}(t)\right]\left(y(t) - \hat{x}(t)\right).$$

B. Proofs of Results in Section V

We begin with a Lemma from [17]:

Lemma A.1: If ν is a p.e. signal satisfying $\dot{\nu}, \nu \in L_\infty$ and $H(s)$ is a stable, minimum phase, proper rational transfer function, then $\nu_2 = H(s)\nu$ is p.e. and $\dot{\nu}_2, \nu_2 \in L_\infty$

Then the proof of Proposition 5.1 is a direct consequence of Lemma A.1 and Proposition 4.1.

Proof of Proposition 5.2: Boundedness of $\bar{q}(t)$ follows from the boundedness of $q_2(t)$, (V5) and the fact that $\|\dot{x}(t)\| \leq \bar{\varepsilon}$, for all t. To prove that $\bar{q}(t)$ is p.e. we need to prove that there exist positive β_i and T_q such that the following holds for all t and $\theta \in \mathbb{R}^n$ with $\|\theta\| = 1$:

$$\beta_1 \leq \int_t^{t+T_q}\left(\theta^\top\left(q_2(\tau) + q(\tau)\right)\right)^2 d\tau \leq \beta_2. \quad \text{(B1)}$$

The upper bound follows from the boundedness of $q_2 + q$. Since $q_2(t)$ is p.e. there exist positive β_3, β_4 and T_2 such that for all t and $\theta \in \mathbb{R}^n$ with $\|\theta\| = 1$ there holds

$$\beta_3 \leq \int_t^{t+T_2}\left(\theta^\top q_2(\tau)\right)^2 d\tau \leq \beta_4. \quad \text{(B2)}$$

Because of (V5) and the fact that $\|\dot{x}(t)\| \leq \varepsilon$, for all t, one also has that $\|q(t)\| \leq \varepsilon/\alpha$ for all t. Thus, we have

$$\begin{aligned}
&\int_t^{t+T_2}\left(\theta^\top\left(q_2(\tau) + q(\tau)\right)\right)^2 d\tau \\
&\geq \int_t^{t+T_2}\left(\theta^\top q_2(\tau)\right)^2 d\tau - \int_t^{t+T_2}\left(\theta^\top q(\tau)\right)^2 d\tau \\
&\geq \beta_3 - T_2\varepsilon^2/\alpha^2. \quad \text{(B3)}
\end{aligned}$$

So the result follows by setting $\beta_1 = \beta_3 - \bar{\varepsilon}^2 T_2/\alpha^2$ and $T_q = T_2$ and by requiring $\bar{\varepsilon} < \sqrt{\beta_3\alpha^2/T_2}$.

C. Proof of Propositions 6.1 and 7.1

To prove Proposition 6.1 we need the following lemma.

Lemma A.2: Consider an L_2 function $g : [0,\infty) \to \mathbb{R}$ and a bounded function $h : [0,\infty) \to \mathbb{R}^n$. Then if $\alpha > 0$ the signal $\rho : [0,\infty) \to \mathbb{R}^n$ defined below

$$\dot{\rho}(t) = -\left(\alpha - g(t)\right)\rho(t) + g(t)h(t) \quad \text{(C1)}$$

with arbitrary $\rho(0)$, is bounded and in L_2.

Proof: First consider the homogeneous system

$$\dot{\rho}_1(t) = -(\alpha - g(t))\,\rho_1(t). \tag{C2}$$

Observe with $u(t) = \rho_1^2(t)/2$, there holds

$$\dot{u}(t) = -(\alpha - g(t))\,u(t) \le -(\alpha - |g(t)|)\,u(t). \tag{C3}$$

By the Bellman–Gronwall inequality for all t_0 and $t \ge t_0$, we have that

$$u(t) \le u(t_0)e^{-\int_{t_0}^{t}(\alpha - |g(s)|)ds}. \tag{C4}$$

Since $g \in L_2$, for every $\epsilon_1 > 0$, there is a t_0, such that for all $t > t_0$

$$\int_{t_0}^{t} g^2(s)ds < \epsilon_1.$$

Consequently, for $t \ge 1 + t_0$, by the Cauchy-Schwarz inequality there holds

$$\int_{t_0}^{t} |g(s)|\,ds < \sqrt{(t-t_0)\int_{t_0}^{t} g^2(s)ds} < \sqrt{(t-t_0)\epsilon_1} \le \sqrt{\epsilon_1}(t-t_0).$$

Thus by choosing $\sqrt{\epsilon_1} < \alpha$, one finds from (C4), that (C2) is exponentially stable, and the result follows because $gh \in L_2$. ∎

Proof of Proposition 6.1: Define, $W(t) = w(t) + q_2(t)$. First consider the last two equations in (V9). For some $a_1 > 0$ to be specified presently, choose the Lyapunov function

$$Q(t) = \tilde{x}^\top(t)\tilde{x}(t) + a_1 p^2(t). \tag{C5}$$

Then there holds

$$\begin{aligned}\dot{Q}(t) =& 2\left(\tilde{x}^\top(t)\dot{\tilde{x}}(t) + a_1 p(t)\dot{p}(t)\right)\\ &- 2\gamma\left(\tilde{x}^\top(t)W(t)\right)^2\\ &- 2\gamma\left(\tilde{x}^\top(t)W(t)\right)p(t) - \alpha a_1 p^2(t)\\ =& -2\gamma\left(\left(\tilde{x}^\top(t)W(t)\right)^2 + \left(\tilde{x}^\top(t)W(t)\right)p(t)\right.\\ &\left. + \frac{\alpha a_1}{2\gamma}p^2(t)\right).\end{aligned} \tag{C6}$$

Thus as long as $a_1\alpha < \gamma/2$, one has

$$\dot{Q}(t) \le -2\gamma\left(\tilde{x}^\top(t)W(t) + \frac{p(t)}{2}\right)^2. \tag{C7}$$

Hence

$$\tilde{x}^\top(t)W(t) + \frac{p(t)}{2} \in L_2. \tag{C8}$$

Further from the third equation in (V9), $p(t)$ converges exponentially to zero. Hence for every β

$$\tilde{x}^\top(t)W(t) + \beta p(t) \in L_2. \tag{C9}$$

Observe from (V3) and (V9)

$$\dot{w}(t) = -\alpha w(t) - (w(t) + q_2(t))\left(\tilde{x}^\top(t)W(t) + \frac{p(t)}{2}\right). \tag{C10}$$

Thus from Lemma A.2, identifying, w, $\tilde{x}^\top(t)W(t) + p(t)/2$, and $q_2(t)$ with ρ, g and h, respectively, $w(t)$ is bounded and in L_2. As q_2 is bounded W must be bounded. As q_2 is p.e. and w is in L_2, $W(t)$ is p.e. as well, [14]. Now observe

$$\dot{\tilde{x}}(t) = -\gamma W(t)W^\top(t)\tilde{x}(t) - \gamma W(t)p(t). \tag{C11}$$

From [13] we know that

$$\dot{\tilde{x}}_1(t) = -\gamma WW^\top\tilde{x}_1(t)$$

is EAS because $W(t)$ is pe. Hence, as $p(t)$ converges exponentially to zero, so does $\tilde{x}(t)$. Exponential convergence of $w(t)$ now follows directly from the first equation in (V9).

Proof of Proposition 7.1: First we rewrite $F(\xi, t)$ as

$$\begin{bmatrix} -\alpha w(t) - \gamma\bar{W}(t)\bar{W}(t)^\top\tilde{x} - \gamma\bar{W}(t)p(t)\\ -\gamma\bar{W}(t)\bar{W}(t)^\top\tilde{x}(t) - \gamma\bar{W}(t)p(t)\\ -\alpha p(t)\end{bmatrix} \tag{C12}$$

where $\bar{q} = q_2 + q$ and $\bar{W}(t) = w(t) + \bar{q}(t)$. Observe $F(\xi, t)$ is identical to the right hand side of (V9) with q_2 replaced by $\bar{q}$. Thus, from Theorem 6.1 and the fact that because of Proposition 5.2, for sufficiently small ε, $\bar{q}(t)$ is bounded and pe, $\dot{\bar{\xi}}(t) = F(\bar{\xi}, t)$ is exponentially asymptotic stable.

The associated convergence parameters, (see remark 6.1) depend only on the p.e. parameters of $\bar{q}$, and because of remark 5.1 bounds on them can be chosen independent of ε.

Hence there exists a Lyapunov function $\mathcal{L}$ and positive real constants c_1, c_2, c_3, and c_4 exist such that [15]

$$c_1\|\xi\|^2 \le \mathcal{L}(\xi, t) \le c_2\|\xi\|^2 \tag{C13}$$

$$\frac{\partial\mathcal{L}}{\partial t} + \frac{\partial\mathcal{L}}{\partial\xi}F(\xi, t) \le -c_3\|\xi\|^2 \tag{C14}$$

$$\left\|\frac{\partial\mathcal{L}}{\partial\xi}\right\| \le c_4\|\xi\|. \tag{C15}$$

Further as the convergence parameters are independent of ε so are the c_i. What is more, $F(\xi, t)$ depends on x only through $\tilde{x}$. Thus the c_i are independent of x as well.

Observe also that there exists a constant K_4 such that for all t

$$\|g_1(\xi, t)\| \le K_4\varepsilon. \tag{C16}$$

Further because of the bound on $y(t) - \hat{x}(t)$, and the fact that $\|q(t)\| \le \bar{\varepsilon}/\alpha$, there is a K_5 independent of ε, such that for all t

$$\|g_2(\xi, t)\| \le K_5\varepsilon. \tag{C17}$$

Now consider

$$\begin{aligned}\dot{\mathcal{L}}(\xi, t) =& \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial\mathcal{L}}{\partial\xi}\left[F(\xi, t) + g_1(\xi, t) + g_2(\xi, t)\right]\\ \le& -c_3\|\xi\|^2 + K_4c_4\varepsilon\|\xi\|^2 + K_5c_4\varepsilon\|\xi\|\end{aligned}.$$

Choose $\bar{\varepsilon} < c_3/(K_4 c_4)$ and call $\beta = c_3 - K_4 c_4 \bar{\varepsilon} > 0$. Observe β is independent of ε. Then there holds

$$\begin{aligned}\dot{\mathcal{L}}(\xi,t) &\leq -\beta\|\xi\|^2 + K_5 c_4 \varepsilon\|\xi\| \\ &\leq -\frac{\beta}{c_2}\mathcal{L}(\xi,t) + c_4 K_5 \varepsilon \sqrt{\frac{\mathcal{L}(\xi,t)}{c_1}}.\end{aligned}$$

Observe if $\sqrt{\mathcal{L}(\xi,t)}$ exceeds $(c_2 c_4 K_5/\beta\sqrt{c_1})\varepsilon$ then $\dot{\mathcal{L}}(\xi,t) < 0$. Thus, arguing as in [16] $\sqrt{\mathcal{L}(\xi,t)}$ is ultimately bounded by $(c_2 c_4 K_5/\beta\sqrt{c_1})\varepsilon$. Consequently $\|\xi\|$ is ultimately bounded by $(c_2 c_4 K_5/\beta c_1)\varepsilon$. As c_i, K_i, and β are independent of ε and the initial time, the result follows.

D. Proof of Theorem 8.1

We need three Lemmas.

Lemma A.3: Consider (VIII2). Suppose for any t_0, all $t \in [t_0, t_0 + (\pi/2|a|)]$, some $\theta \in \mathbb{R}^2$, there exists ϵ' such that $|\theta^\top \dot{\zeta}(t)| \leq \epsilon'\|\theta\|$. Then

$$\epsilon' \geq |a|\|\zeta(t_0)\|\|\theta\|. \tag{D1}$$

Further with $\xi = [\xi_1, \xi_2]^\top$, for all t_0 and $i \in \{1,2\}$

$$\left|\zeta_i\left(t_0 + \frac{\pi}{2|a|}\right) - \zeta_i(t_0)\right| = \|\zeta(t_0)\|. \tag{D2}$$

Proof: For some real ψ there holds $\theta = \|\theta\|[\sin\psi, -\cos\psi]^\top$. Hence, $E^\top\theta = \|\theta\|[\cos\psi, \sin\psi]^\top$. Thus under (VIII2)

$$\theta^\top E\zeta(t) = \|\xi(t_0)\|\,\|\theta\|\cos(a(t-t_0) + \beta(t_0) - \psi). \tag{D3}$$

Therefore, on any interval $[t_0, t_0 + \pi/2|a|]$ the maximum of $|\theta^\top\dot{\xi}| = |a\theta^\top E\xi|$ is $|a|\|\zeta(t_0)\|\|\theta\|$. Further (D2) is a direct consequence of (VIII2). ∎

Next we prove the following lemma.

Lemma A.4: Consider

$$\dot{\zeta}(t) = af(t)E\zeta(t) \tag{D4}$$

where $f: \mathbb{R} \to \mathbb{R}$ and $|f(t)| \leq 1\ \forall t$. Then for all $0 \leq t_0 \leq t$, $\|\zeta(t) - \zeta(t_0)\| \leq (t-t_0)|a|\|\zeta(t_0)\|$.

Proof: Under (D4), for all $t \geq t_0$, $\|\zeta(t)\| = \|\zeta(t_0)\|$. Thus

$$\begin{aligned}\|\xi(t) - \xi(t_0)\| &= \left\|\int_{t_0}^{t} f(\tau)aE\zeta(\tau)d\tau\right\|. \\ &\leq (t-t_0)|a|\,\|\zeta(t_0)\|\end{aligned}$$

∎

Lastly, we present the following result from [18].

Lemma A.5: Suppose on a closed interval $\mathcal{I} \subset \mathbb{R}$ of length Ω, a signal $u: \mathcal{I} \to \mathbb{R}$ is twice differentiable and for some ϵ_1 and M'

$$|u(t)| \leq \epsilon_1 \text{ and } |\ddot{u}(t)| \leq M' \quad \forall t \in \mathcal{I}.$$

Then for some M independent of ϵ_1, $\mathcal{I}$ and M', and $M'' = \max(M', 2\epsilon_1\Omega^{-2})$ one has

$$|\dot{u}(t)| \leq M(M''\epsilon_1)^{1/2} \quad \forall t \in \mathcal{I}.$$

Proof of Theorem 8.1: We will prove the result by contradiction. First observe that as $A(t)$ is differentiable and $\ddot{y}^*$ is bounded. Also observe that if (III13) holds for $\|y^*(0)\| = 1$, then it holds for arbitrary $\|y^*(0)\|$. Thus assume that $\|y^*(0)\| = 1$. Consequently for all $t \geq 0$

$$\|y^*(t)\| = 1. \tag{D5}$$

Suppose (III13) is violated. Then for all $\epsilon_2 > 0$ and $T_3 > 0$, there exists a t_0 and a unit norm $\theta = [\theta_1, \theta_2, \theta_3]^\top \in \mathbb{R}^3$, such that

$$\int_{t_0}^{t_0+T_3} \left(\theta^\top \dot{y}^*(\tau)\right)^2 d\tau \leq \epsilon_2^2.$$

Thus from Lemma A.5 for some M_3, all $\epsilon_2 > 0$, some $T_4(\epsilon_2)$, dependent only on the bound on $\ddot{y}^*(\cdot)$ and ϵ_2, and all $T_3 > T_4(\epsilon_2)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$\left|\theta^\top \dot{y}^*(t)\right| \leq M_3\epsilon_2^{1/2} \quad \forall t \in [t_0, t_0 + T_3]. \tag{D6}$$

Choose

$$t_1 = \min\{kT \geq t_0 + T_4(\epsilon_2) | k \in \mathbb{Z}_+\}. \tag{D7}$$

Denote $y^* = [y_1^*, y_2^*, y_3^*]^\top$. Observe at least one of $\|[\theta_1, \theta_2]^\top\|$ or $\|[\theta_2, \theta_3]^\top\|$ must exceed $1/\sqrt{3}$, since θ has unit norm. We consider two cases.

Case I: $\|[\theta_1, \theta_2]^\top\| > 1/\sqrt{3}$. Since the inequality in (D6) holds on the indicated interval, it must hold for all $t \in [t_1 + kT + \bar{T}_4, t_1 + kT + \bar{T}_5]$, $k \in \mathbb{Z}$. Thus for all $t \in [t_1 + kT + \bar{T}_4, t_1 + kT + \bar{T}_5]$ and $k \in \mathbb{Z}$, there holds

$$\left|[\theta_1, \theta_2]^\top [\dot{y}_1^*(t), \dot{y}_2^*(t)]\right| \leq M_3\epsilon_2^{1/2}. \tag{D8}$$

Now for all $t \in [t_1 + kT + \bar{T}_4, t_1 + kT + \bar{T}_5]$, there also holds $[\dot{y}_1^*(t)\ \dot{y}_2^*(t)]^\top = cE[y_1^*(t)\ y_2^*(t)]^\top$. Thus, from (D1) of Lemma A.3 and the hypothesis of the case, we obtain that for all $k \in \mathbb{Z}$

$$\left\|\left[y_1^*(t_1 + kT + \bar{T}_4), y_2^*(t_1 + kT + \bar{T}_4)\right]^\top\right\| \leq \frac{\sqrt{3}M_3}{|c|}\epsilon_2^{1/2}. \tag{D9}$$

Further with some $h_1: \mathbb{R} \to \mathbb{R}$, in the interval $[kT + \bar{T}_4, (k+1)T]$, $[\dot{y}_1^*(t)\ \dot{y}_2^*(t)]^\top = h_1(t)E[y_1^*(t)\ y_2^*(t)]^\top$. Thus

$$\begin{aligned}&\left\|\left[y_1^*(t_1 + (k+1)T), y_2^*(t_1 + (k+1)T)\right]^\top\right\| \\ &= \left\|\left[y_1^*(t_1 + kT + \bar{T}_4), y_2^*(t_1 + kT + \bar{T}_4)\right]^\top\right\| \\ &\leq \frac{\sqrt{3}M_3}{|c|}\epsilon_2^{1/2}.\end{aligned} \tag{D10}$$

Consequently because of (D5), there holds

$$\begin{aligned}&\left\|\left[y_2^*(t_1 + (k+1)T), y_3^*(t_1 + (k+1)T)\right]^\top\right\| \\ &\geq |y_3^*(t_1 + (k+1)T)| \\ &\geq 1 - \frac{\sqrt{3}M_3}{|c|}\epsilon_2^{1/2}.\end{aligned} \tag{D11}$$

Further throughout the interval $t \in [t_1 + kT, t_1 + kT + T_3]$ for some $h_2 : \mathbb{R} \to \mathbb{R}$, $|h_2(t)| \leq 1$

$$[\dot{y}_2^*(t) \quad \dot{y}_3^*(t)]^\top = h_2(t) E \, [y_2^*(t) \quad y_3^*(t)]^\top . \tag{D12}$$

Thus from Lemma A.4 and (D10)

$$\left| y_2^*(t_1 + kT + \bar{T}_1) \right| \leq \frac{\sqrt{3} M_3}{|c|} \epsilon_2^{1/2} + \rho |b|. \tag{D13}$$

Also from (D12) and (D11)

$$\left\| [y_2^*(t), y_3^*(t)]^\top \right\| \geq 1 - \frac{\sqrt{3} M_3}{|c|} \epsilon_2^{1/2}$$

holds for all $t \in [t_1 + kT, t_1 + kT + \bar{T}_3]$. Notice in the interval $[[t_1 + kT + \bar{T}_1, t_1 + kT + \bar{T}_2]$, (D12) holds with $h_2(t) = b$. Thus from (D2) of Lemma A.3

$$\begin{aligned} &\left| y_2^*(t_1 + kT + \bar{T}_2) \right| + \left| y_2^*(t_1 + kT + \bar{T}_1) \right| \\ &\quad \geq \left| y_2^*(t_1 + T + \bar{T}_2) - y_2^*(t_1 + T + \bar{T}_1) \right| \\ &\quad \geq 1 - \frac{\sqrt{3} M_3}{|c|} \epsilon_2^{1/2}. \end{aligned} \tag{D14}$$

Consequently, from (D13)

$$\left| y_2^*(t_1 + kT + \bar{T}_2) \right| \geq 1 - \frac{2\sqrt{3} M_3}{|c|} \epsilon_2^{1/2} - \rho |b|.$$

Further, from Lemma A.4

$$\left| y_2^*(t_1 + kT + \bar{T}_4) \right| \geq 1 - \frac{2\sqrt{3} M_3}{|c|} \epsilon_2^{1/2} - \rho \left(2|b| + |c| \right). \tag{D15}$$

Then for

$$\rho < \frac{1}{4 \left(|b| + |c| \right)} \tag{D16}$$

and sufficiently small ϵ_2, (D15), contradicts with (D9).

Case II: $\|[\theta_2, \theta_3]^\top\| > 1/\sqrt{3}$. Follows similarly with the same set of ρ given in (D16).

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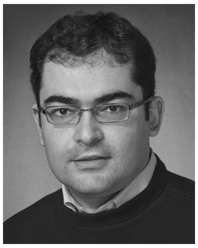
Iman Shames (M'11) received the B.S. degree in electrical engineering from Shiraz University, Shiraz, Iran, in 2006, and the Ph.D. degree in engineering and computer science from the Australian National University, Canberra, Australia, in 2010.

He is currently a Postdoctoral Researcher at the ACCESS Linnaeus Centre, the Royal Institute of Technology (KTH), Stockholm, Sweden. He has been a Visiting Researcher at the University of Tokyo, Tokyo, Japan, in 2008, and at the University of Newcastle, Newcastle, Australia, in 2005. His current research interests include multi-agent systems, sensor networks, systems biology, and distributed fault detection and isolation.

Soura Dasgupta (F'98) was born in Calcutta, India, in 1959. He received the B.E. degree in electrical engineering from the University of Queensland, Brisbane, Australia, in 1980, and the Ph.D. in systems engineering from the Australian National University, Canberra, Australia, in 1985.

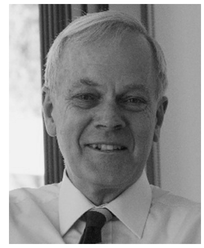
He is currently Professor of Electrical and Computer Engineering at the University of Iowa, Iowa City. In 1981, he was a Junior Research Fellow in the Electronics and Communications Sciences Unit, Indian Statistical Institute, Calcutta. He has held visiting appointments at the University of Notre Dame, University of Iowa, Université Catholique de Louvain-La-Neuve, Belgium, National ICT Australia, and the Australian National University. He is a past Presidential Faculty Fellow, a Subject Editor for the *International Journal of Adaptive Control and Signal Processing*, and a member of the editorial board of the *EURASIP Journal of Wireless Communications*. His research interests are in controls, signal processing and communications.

Dr. Dasgupta received the Gullimen Cauer Award for the best paper published in the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS in the calendar years of 1990 and 1991, Between 1988 and 1991, 2004 and 2007 and 1999 to 2009, he respectively served as an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS-II, and the IEEE Control Systems Society Conference Editorial Board.



Barış Fidan (SM'11) received the B.S. degrees in electrical engineering and mathematics from the Middle East Technical University, Ankara, Turkey, in 1996, the M.S. degree in electrical engineering from Bilkent University, Ankara, Turkey, in 1998, and the Ph.D. degree in electrical engineering from the University of Southern California, Los Angeles, in 2003.

He worked at the University of Southern California in 2004 as a Postdoctoral Research Fellow, and at the National ICT Australia and the Research School of Information Sciences and Engineering, Australian National University, from 2005 to 2009, as Researcher/Senior Researcher. He is currently an Assistant Professor at the Mechanical and Mechatronics Engineering Department, University of Waterloo, Waterloo, ON, Canada. His research interests include autonomous multi-agent dynamical systems, sensor networks, cooperative target localization, adaptive and nonlinear control, switching and hybrid systems, mechatronics, and various control applications including vehicle and transportation control, high performance and hypersonic flight control, semiconductor manufacturing process control, and disk-drive servo systems.



Brian D. O. Anderson (LF'06) was born in Sydney, Australia. He received the B.S. and M.S. degrees in pure mathematics and electrical engineering from the University of Sydney, Sydney, Australia and the PhD degree in electrical engineering from Stanford University, Stanford, CA.

Following completion of his education, he worked in industry in Silicon Valley and served as a faculty member in the Department of Electrical Engineering at Stanford. He was Professor of Electrical Engineering at the University of Newcastle, Newcastle, Australia from 1967 until 1981 and is now a Distinguished Professor at the Australian National University and Distinguished Researcher in National ICT Australia Ltd. His interests are in control and signal processing.

Dr. Anderson received the IEEE Control Systems Award of 1997, the 2001 IEEE James H. Mulligan, Jr. Education Medal, and the Guillemin-Cauer Award, IEEE Circuits and Systems Society in 1992 and 2001, the Bode Prize of the IEEE Control System Society in 1992, and the Senior Prize of the IEEE Transactions on Acoustics, Speech and Signal Processing in 1986. He is a Fellow of the Royal Society London, Australian Academy of Science, Australian Academy of Technological Sciences and Engineering, Honorary Fellow of the Institution of Engineers, Australia, and Foreign Associate of the U.S. National Academy of Engineering. He holds doctorates (honoris causa) from the Université Catholique de Louvain, Belgium, Swiss Federal Institute of Technology, Zurich, Universities of Sydney, Melbourne, New South Wales and Newcastle. He served a term as President of the International Federation of Automatic Control from 1990 to 1993 and as President of the Australian Academy of Science between 1998 and 2002.