Distance Estimation From Received Signal Strength Under Log-Normal Shadowing: Bias and Variance

Sree Divya Chitte, Soura Dasgupta, *Fellow, IEEE*, and Zhi Ding, *Fellow, IEEE*

***Abstract*—In source localization, one estimates the location of a source using a variety of relative position information. Many algorithms use certain powers of distances to effect localization. In practice, exact distance measurement is not directly available and must be estimated from information such as received signal strength (RSS), time of arrival, or time difference of arrival. This letter considers bias and variance issues in estimating powers of distances from RSS affected by practical log-normal shadowing. We show that the underlying estimation problem is inefficient and that the maximum likelihood estimate yields a bias and a mean-square error (MSE) that both increase exponentially with the noise power. We then characterize the class of unbiased estimates and show that there is *only one estimator in this class*, but that its MSE also grows exponentially with the noise power. Finally, we provide the linear minimum mean-square error (MMSE) estimate and show that its bias and MSE are both bounded in the noise power.**

***Index Terms*—Localization, maximum likelihood, received signal strength, sensors, unbiased.**

I. Introduction

The last few years have witnessed significant increase of research activity in the area of source localization [1]. Localization involves a group of sensors jointly estimating the location of a signal source using such relative position information as distance, bearing, received signal strength (RSS), time of arrival (TOA), and time difference of arrival (TDOA).

Localization is fundamental to a number of emerging applications [1]. For example, a network of sensors deployed to combat bioterrorism must not only detect the presence of a potential threat but must also locate its source. In (wireless) pervasive computing [2]–[4], localization enables the computer network to identify the most appropriate serving units with matching capabilities for the users. In sensor networks [5], individual sensors must know their own positions, to route packets, detect faults, and detect and record events. As compellingly, [6] catalogs a burgeoning multibillion dollar market surrounding wireless location technology. A common source of relative position information is the RSS. In particular, suppose a source emits a signal that has strength A at a unit distance from the source. Suppose the signal strength at a distance d from the source is s. Then with β as the path loss coefficient, in the absence of noise, one has

$$s = \frac{A}{d^{\beta}}. \tag{1}$$

In the noise-free case, s directly provides the distance d, provided β and A are known. Indeed in the sequel, we will assume the knowledge of these two parameters. Several papers such as [6]–[14] present localization algorithms that assume that powers of distances are known. The distance itself is rarely directly available, but it must be deduced from information such as RSS, TOA, or TDOA. At the same time, in far field, RSS is usually affected by log-normal shadowing [15], i.e., with $w \sim N(0, \sigma^2)$, (1) must be replaced by

$$\ln s = \ln A - \beta \ln d + w. \tag{2}$$

There are papers in the literature, e.g., [17] and [18], that study the accuracy of localization using RSS measurements.

This letter considers the estimation of d^m for some positive m from s when (2) applies. The focus is on the issues of bias and mean-square error (MSE). Indeed as argued in Section II, this estimation problem is inefficient in that it has no unbiased estimate that meets the Cramer–Rao lower bound (CRLB), thus motivating the study of this problem. A further motivation comes from the fact also demonstrated in this section that not only is the maximum likelihood estimate (MLE) biased but also has bias and MSE that grow exponentially with σ^2.

Thus, in Section III, we consider the nature of unbiased estimators. Using techniques developed in the literature on complete sufficient statistics of exponential family of distributions [16], we show that there is in fact a *unique unbiased estimator* and that its variance also grows *exponentially with* σ^2. Finally, in Section IV, we provide the linear minimum mean-square error (MMSE) estimator whose MSE and bias are both shown to be bounded in σ^2.

II. Preliminaries

For some positive real m, define

$$\alpha = \frac{m}{\beta}.$$

Then with $p = d^m$ and

$$z = \left(\frac{A}{s}\right)^{\alpha} \tag{3}$$

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S. D. Chitte and S. Dasgupta are with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242 USA (e-mail: schitte@engineering.uiowa.edu; dasgupta@engineering.uiowa.edu).

Z. Ding is with the Department of Electrical and Computer Engineering, University of California, Davis, CA 95616 USA (e-mail: zding@ucdavis.edu).

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(2) becomes

$$z = e^{-\alpha w} p. \tag{4}$$

The underlying estimation problem is to estimate p from the observation z, obeying (4) and the knowledge of α and σ^2.

Consider now the derivation of the CRLB for this problem. Observe that (4) is equivalent to

$$\ln z = \ln p - \alpha w. \tag{5}$$

Call $y = \ln z$, and observe that

$$y \sim N(\ln p, \alpha^2\sigma^2).$$

The log-likelihood function is given by

$$l(y,p) = -\ln\left[\sqrt{2\pi}\sigma\alpha\right] - \frac{(y-\ln p)^2}{2\alpha^2\sigma^2}.$$

As

$$\frac{\partial l(y,p)}{\partial p} = \frac{y-\ln p}{\alpha^2\sigma^2 p}$$

we have that

$$E\left[\left(\frac{\partial l(y,p)}{\partial p}\right)^2\right] = \frac{1}{p^2\alpha^2\sigma^2}.$$

Thus, for this problem

$$\mathrm{CRLB} = p^2\alpha^2\sigma^2 \tag{6}$$

which increases linearly with σ^2.

One then asks whether an efficient estimator exists for this problem, i.e., is there an unbiased estimate whose MSE matches the CRLB. To this end, consider (5) and observe that the data have an affine dependence on the Gaussian noise w but a *non-affine dependence on* p. Thus, from a result in [19], we conclude that no efficient estimate of p exists.

This leads us to examine the properties of MLE, which from (5) equals

$$p_{\mathrm{ML}} = z. \tag{7}$$

From (4) and the fact that for any a

$$E[e^{aw}] = e^{a^2\sigma^2/2} \tag{8}$$

one obtains the bias

$$E[p_{\mathrm{ML}}] - p = E\left[pe^{-\alpha w}\right] - p = \left(e^{\alpha^2\sigma^2/2} - 1\right)p.$$

Further, the MSE is given by

$$\begin{aligned} E\left[(p_{\mathrm{ML}} - p)^2\right] &= E[(e^{-\alpha w} - 1)^2]p^2 \\ &= E[e^{-2\alpha w} - 2e^{-\alpha w} + 1]p^2 \\ &= \left(e^{2\alpha^2\sigma^2} - 2e^{\alpha^2\sigma^2/2} + 1\right)p^2. \end{aligned}$$

Thus, both the bias and the MSE of the MLE *grow exponentially with* σ^2.

At the same time, we note that MLE is asymptotically efficient. Thus, in settings where the MLE is obtained using multiple measurements, an improved performance will be noted.

III. Best Unbiased Estimate

Given that there are no efficient estimators for (4), it behooves us to determine the best unbiased estimator for this problem. To this end, consider an arbitrary estimator $f(z)$ whose mean is p for all $p > 0$, i.e.,

$$E[f(z)] = p. \tag{9}$$

As α and σ^2 are known, we permit $f(\cdot)$ to be a function of α and σ^2. Observe this requires that for all $p > 0$, there hold

$$p = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} f(pe^{-\alpha w})e^{-\frac{w^2}{2\sigma^2}}dw. \tag{10}$$

Then because of (5), we have that for all $p > 0$, there holds

$$\frac{1}{\sqrt{2\pi}\sigma\alpha}\int_0^{\infty}\frac{f(z)}{z}\exp\left(-\frac{(\ln z - \ln p)^2}{2\alpha^2\sigma^2}\right)dz = p. \tag{11}$$

Now define

$$t = \ln z \tag{12}$$

and

$$v = \frac{\ln p}{\alpha^2\sigma^2}. \tag{13}$$

Then for all v, (11) becomes

$$e^{\alpha^2\sigma^2 v} = \frac{1}{\sqrt{2\pi}\sigma\alpha}\int_{-\infty}^{\infty} f(e^t)e^{-\frac{t^2}{2\alpha^2\sigma^2}}e^{vt}dte^{-\frac{v^2\alpha^2\sigma^2}{2}}$$

i.e., for all v, there holds

$$\exp\left(\alpha^2\sigma^2 v + \frac{v^2\alpha^2\sigma^2}{2}\right) = \frac{1}{\sqrt{2\pi}\sigma\alpha}\int_{-\infty}^{\infty} f(e^t)e^{-\frac{t^2}{2\alpha^2\sigma^2}}e^{vt}dt. \tag{14}$$

Thus, with v and t, the two domain variables

$$\exp\left(\alpha^2\sigma^2 v + \frac{v^2\alpha^2\sigma^2}{2}\right) \tag{15}$$

and

$$\frac{1}{\sqrt{2\pi}\sigma\alpha}f(e^t)e^{-\frac{t^2}{2\alpha^2\sigma^2}}$$

are Laplace pairs. *This clearly proves that* $f(z)$ *is unique*.

Further, (15) is nothing more than the moment generating function of

$$N(\alpha^2\sigma^2, \alpha^2\sigma^2).$$

Thus, the uniqueness of Laplace pairs establishes the following:

$$\frac{1}{\sqrt{2\pi}\sigma\alpha}f(e^t)e^{-\frac{t^2}{2\alpha^2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma\alpha}\exp\left(-\frac{(t-\alpha^2\sigma^2)^2}{2\alpha^2\sigma^2}\right)$$

from which we obtain that

$$f(e^t) = e^{-\frac{\alpha^2\sigma^2}{2}} e^t.$$

Thus, one has that the *only unbiased estimate* is given by

$$p_u = e^{-\frac{\alpha^2\sigma^2}{2}} z. \tag{16}$$

We now examine the MSE of p_u. There holds

$$\begin{aligned} E[(p_u - p)^2] &= E\left[\left(e^{-\frac{\alpha^2\sigma^2}{2}} e^{-\alpha w} - 1\right)^2\right] p^2 \\ &= E\left[e^{-\alpha^2\sigma^2} e^{-2\alpha w} - 2e^{-\frac{\alpha^2\sigma^2}{2}} e^{-\alpha w} + 1\right] p^2 \\ &= \left[e^{-\alpha^2\sigma^2} e^{2\alpha^2\sigma^2} - 2e^{-\frac{\alpha^2\sigma^2}{2}} e^{\frac{\alpha^2\sigma^2}{2}} + 1\right] p^2 \\ &= \left(e^{\alpha^2\sigma^2} - 1\right) p^2. \end{aligned}$$

Thus, this MSE too rises exponentially with σ^2.

IV. Linear MMSE Estimate

The previous sections show that MLE has a bias that grows exponentially with σ^2, as do the MSEs of MLE and the only unbiased estimate. Contrast this to the fact that the CRLB grows linearly with σ^2.

Recall also that the unbiased estimate of Section III is in fact linear in z. Thus, we derive the linear MMSE estimate, linearity being in the observation z. We wish to find a b that minimizes

$$E[(bz - p)^2] = E[(be^{-\alpha w} - 1)^2]p^2. \tag{17}$$

Clearly, the minimizing b obeys

$$b = \frac{E[e^{-\alpha w}]}{E[e^{-2\alpha w}]} = e^{\frac{-3\alpha^2\sigma^2}{2}}. \tag{18}$$

Thus, the estimate we seek is

$$p_v = e^{\frac{-3\alpha^2\sigma^2}{2}} z. \tag{19}$$

Its bias is

$$\begin{aligned} E(p_v) - p &= \left(e^{-\frac{3\alpha^2\sigma^2}{2}} E[e^{-\alpha w}] - 1\right) p \\ &= \left(e^{-\alpha^2\sigma^2} - 1\right) p. \end{aligned}$$

Likewise, the MSE is $E[(p_v - p)^2]$

$$\left(e^{-3\alpha^2\sigma^2} E[e^{-2\alpha w}] - 2e^{\frac{-3\alpha^2\sigma^2}{2}} E[e^{-\alpha w}] + 1\right) p^2 \\ = \left(1 - e^{-\alpha^2\sigma^2}\right) p^2.$$

Observe unlike p_{ML} or p_u whose MSEs grow exponentially with σ^2, or for that matter the CRLB which grows linearly in σ^2, the MSE of p_v is bounded by p^2. Likewise, the bias is also bounded by p in magnitude. Thus, in fact, for large noise power, this MSE is better than the CRLB. This is of course not a surprise, as the estimate p_v is biased.

It is also noteworthy that while the bias in MLE is always positive, that in p_v is always negative.

V. Conclusion

We have considered the estimation of d^m from RSS when the latter is corrupted by log-normal shadowing. We have shown that the underlying estimation problem is inefficient and that both the bias and the MSE of MLE grow exponentially with the noise power. We have also demonstrated that there is a unique unbiased estimator whose MSE also grows exponentially with the noise power. Finally, we have provided the linear MMSE estimator for which both the magnitude of the bias and the MSE are bounded in the noise power. The implication of these facts to source localization directly will be the subject of a forthcoming paper.

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