Reduced Complexity Semidefinite Relaxation Algorithms for Source Localization Based on Time Difference of Arrival

Enyang Xu, *Student Member*, *IEEE*, Zhi Ding, *Fellow*, *IEEE*, and Soura Dasgupta, *Fellow*, *IEEE*

Abstract—We investigate the problem of source localization based on measuring time difference of signal arrivals (TDOA) from the source emitter. Taking into account the colored measurement noise, we adopt a min-max principle to develop two lower complexity semidefinite relaxation algorithms that can be reliably solved using semidefinite programming. The reduction of algorithm complexity is achieved through a simple, but effective method to select a reference node among participating measurement nodes such that only selective time differences of signal arrival are exploited. Our estimation methods are insensitive to the source locations and can be used either as the final location estimate or as the initial point for more traditional search algorithms.

Index Terms—Source localization, time difference of arrival, semidefinite programming.

1 Introduction

Wireless source localization has been a problem that maintains a considerable level of research interest because of its broad applications in areas such as target tracking, signal routing, interference mitigation, and emergency response, among others [2]. Source localization typically involves estimating the positions of signal emitters in a network of sensors that measure distinct source signal characteristics. Utilizing the collective signal measurements from the sensors, a data fusion center can generate the source location estimate. In practice, the various data fusing methods include time of arrival (TOA), time difference of arrival (TDOA), received signal strength (RSS), angle of arrival (AOA), and various combinations of these [3]. Similarly, an equivalent problem of sensor navigation involves a sensor trying to estimate its own location based on signals received from multiple (coordinated) emitters.

The problem of source localization has been investigated in a number of published works, e.g., [2], [3], based on various signal measurement models. The authors of [4] presented a semidefinite programming (SDP) algorithm for a noisy distance measurement (DM) model by minimizing the l^1 norm of estimation error. This proposed framework can also integrate additional angle-of-arrival information. By applying a RSS measurement under the well-known log-normal fading model, the authors of [5] also derived efficient SDP approaches to source localization based on a min-max criteria. However, in most applications, distance, and signal strength measurements are not directly available. Additionally, in environment rich with scatters, radio signal strength measurement can be highly variant and noisy. Therefore, developing algorithms less dependent on source location for alternative measurement models becomes necessary. In this work, we are particularly interested in the TDOA measurement model, for which only differences of the signal arrive instants between different sensor nodes are considered. There is no need to synchronizing the clock between the source and the receiving sensors in the TDOA model.

Typically, preprocessing TDOA measurements will lead to a set of linear equations with some information loss [3]. In [6], the authors proposed a two-step generalized least square (LS) method as an approximate maximum likelihood (ML) solution. On the other hand, the authors of [7] considered LS approaches based on range difference and squared range difference measurements, through which the global optimal solution is found by adding a correction term to the LS inverse matrix. However, this correction term requires a search process involving matrix inverses which may be ill-conditioned. More recently, the authors of [8] provided a rather elegant solution by assuming noise independence among the TDOA measurement, leading to an approximate ML formulation for TDOA based localization. The work of [8] is based on an effective relaxation method to transform the original nonconvex optimization problem into a convex problem. All pairwise TDOA measurements from sensor nodes are included in a cost function for minimization. This means that, with N sensors, the number of pairwise TDOA pairs under consideration is $N(N-1)$, potentially leading to high computational complexity. Similar to [8], the authors in [9] considered only one reference node and proposed a constraint based on prior location knowledge.

In this work, we develop two new algorithms that are derived from a reduced complexity semidefinite relaxation

- *E. Xu and Z. Ding are with the Department of Electrical and Computer Engineering, University of California, Davis, 2064 Kemper Hall, 1 Shields Avenue, Davis, CA 95616. E-mail: {eyxu, zding}@ece.ucdavis.edu.*
- *S. Dasgupta is with the Department of Electrical and Computer Engineering, 5322 Seamans Center for the Engineering Arts and Sciences, University of Iowa, Iowa City, IA 52242. E-mail: dasgupta@engineering.uiowa.edu.*

Manuscript received 28 Feb. 2010; revised 3 Aug. 2010; accepted 3 Sept. 2010; published online 20 Dec. 2010.
For information on obtaining reprints of this article, please send e-mail to: tmc@computer.org, and reference IEEECS Log Number TMC-2010-02-0097.
Digital Object Identifier no. 10.1109/TMC.2010.263.

1536-1233/11/$26.00 © 2011 IEEE Published by the IEEE CS, CASS, ComSoc, IES, & SPS

for the TDOA model. In particular, we first identify a reference node to selectively generate the essential TDOA measurement information. We formulate a simpler approximation of the source localization problem based on the measured TDOA information between the reference node and other nonreference nodes. We develop a min-max criterion that is less sensitive to the intrinsic measurement noise correlation and can be solved via semidefinite relaxations. Thus, our algorithm does not require the assumption on correlation of TDOA measurement noise and is less sensitive to the distribution of the sensors versus the source. We also propose a simple and effective way for the selection of the reference node.

2 Problem Statement

2.1 Time of Arrival Measurement Model

Consider a network that consists of N sensors at known positions denoted by a set of m-dimensional vectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$ (with $m = 2$ or 3). These sensors cooperate to determine an unknown source location denoted by an m-dimensional vector $\mathbf{y}$. Note that we restrict our focus only on propagation environments without remote scatters. In other words, the time of arrival measurement can be approximately modeled by a line of sight environment. Based on collected sensor measurements $\{v_1, v_2, \ldots, v_N\}$, a data fusion center generates a source location estimate

$$\hat{\mathbf{y}}_0 = \phi(v_1, \ldots, v_N; \mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{1}$$

where $\phi(.)$ is the functional form of the estimate.

With sensors synchronized to a common clock, the common source signal arrives at different sensors with TOA values t_i of a specific node $\mathbf{x}_i$

$$t_i = \frac{1}{c}\|\mathbf{x}_i - \mathbf{y}\| + t_0 + n_i, \qquad i = 1, 2, \ldots, N. \tag{2}$$

Here, c is the speed of light and t_0 is the unknown reference time at which the source signal was transmitted. In addition, n_i are independent identically distributed (i.i.d.) Gaussian with zero mean and variance σ^2. Under the i.i.d. Gaussian noise assumption, the conditional probability density of the measurement data is

$$p(t_1, t_2, \ldots, t_N|\mathbf{y}, t_0) = (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(t_i - \frac{1}{c}\|\mathbf{x}_i - \mathbf{y}\| - t_0\right)^2\right). \tag{3}$$

Consequently, the maximum likelihood estimate (MLE) of $\mathbf{y}$ can be obtained for the unprocessed time-of-arrival (TOA) measurement as

$$\hat{\mathbf{y}} = \arg\min_{\mathbf{y},t_0} \sum_{i=1}^{N}\left(t_i - \frac{1}{c}\|\mathbf{x}_i - \mathbf{y}\| - t_0\right)^2. \tag{4}$$

2.2 Estimation from Time Differences of Arrival

Jointly, estimating $\mathbf{y}$ and t_0 accurately can be challenging. However, by designating one of the sensors as a reference node $\mathbf{x}_r, 1 \le r \le N$, we can remove one unknown t_0 and obtain a much simpler model for estimation based on TDOA measurement between the ith node and the reference node

$$\Delta_{i,r} = \frac{1}{c}(\|\mathbf{x}_i - \mathbf{y}\| - \|\mathbf{x}_r - \mathbf{y}\|) + n_i - n_r, \quad i = 1, 2, \ldots, r-1, r+1, \ldots, N. \tag{5}$$

It should be noted that the noise term in (5) is $n_i - n_r$. Therefore, the noise terms in $\Delta_{i,r}$ and $\Delta_{j,r}$ are dependent. For a specific reference node $\mathbf{x}_r$, let

$$\Delta_r = [\Delta_{1,r} \ldots \Delta_{r-1,r}\ \Delta_{r+1,r} \ldots \Delta_{N,r}]^T,$$
$$\mu_{i,r} = \frac{1}{c}(\|\mathbf{x}_i - \mathbf{y}\| - \|\mathbf{x}_r - \mathbf{y}\|),$$
$$\mu_r = [\mu_{1,r} \cdots \mu_{r-1,r}\ \mu_{r+1,r} \ldots \mu_{N,r}]^T.$$

Then, the joint conditional probability density of Δ_r is

$$p(\Delta_r|\mathbf{y}) = (2\pi\sigma^2)^{-\frac{N-1}{2}}|\mathbf{Q}|^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(\Delta_r - \mu_r)^T\mathbf{Q}^{-1}(\Delta_r - \mu_r)\right), \tag{6}$$

where

$$\mathbf{Q} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}.$$

Then, the approximate MLE of $\mathbf{y}$ becomes

$$\hat{\mathbf{y}}_1 = \arg\min_{\mathbf{y}}[(\Delta_r - \mu_r)^T\mathbf{Q}^{-1}(\Delta_r - \mu_r)]. \tag{7}$$

We also note that if we include all pairwise TDOA measurements into consideration like in [8], then the noise correlation leads to a noninvertible covariance matrix whenever $N > 3$ due to measurement reuse. Still, some TDOA source localization approaches, such as [8], neglect this noise correlation. Thus, the resulting localization algorithms based on independent TDOA noise assumption tend to suffer performance degradations when applied in practical TDOA measurement. Therefore the noise correlation must be considered in determining MLE and Cramer-Rao Lower Bound (CRLB).

For a specific reference node, the Fisher information matrix is given by [6] as

$$\mathbf{F} = \frac{1}{c^2\sigma^2}\sum_{i=1}^{N}\frac{(\mathbf{x}_i - \mathbf{y})(\mathbf{x}_i - \mathbf{y})^T}{\|\mathbf{x}_i - \mathbf{y}\|^2} - \frac{1}{Nc^2\sigma^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{(\mathbf{x}_i - \mathbf{y})(\mathbf{x}_j - \mathbf{y})^T}{\|\mathbf{x}_i - \mathbf{y}\|\|\mathbf{x}_j - \mathbf{y}\|}, \tag{8}$$

which is independent of the choice of the reference point $\mathbf{x}_r$. Hence, for any unbiased estimate $\hat{\mathbf{y}}$, the CRLB is

$$\text{MSE} = E(\|\hat{\mathbf{y}} - \mathbf{y}\|^2) \ge \text{Trace}[\mathbf{F}^{-1}]. \tag{9}$$

Note, however, that t_i is not linear with $\mathbf{y}$ in (2). Hence, according to the analysis of [10], there exists no *efficient unbiased* estimators for $\mathbf{y}$ in the TDOA measurement model. Therefore, even though the CRLB is invariant to the choice of reference node $\mathbf{x}_r$, it only implies that the performance of an efficient unbiased estimate, if it exists, is not affected by the reference point selection. Because there exists no efficient unbiased estimator, a given localization algorithm based on TDOA model either is biased, or fails to reach the CRLB. Thus, the performance of actual TDOA localization algorithms will be affected by the reference node selection.

3 A New SDP Relaxation Based on TDOA

3.1 Localization Criterion with Low Sensitivity to Noise Correlation

The direct solution of the MLE $\hat{\mathbf{y}}$ in (7) is a nonconvex optimization problem. In [8], a novel relaxed convex formulation was proposed, in which all the pairwise TDOA measurements are involved. However, its reliance on independence TDOA noise assumption makes it more sensitive to the true TDOA model in which the noises are correlated. In addition, the complexity can be quite high by including all pairwise TDOA measurements.

To develop a simpler TDOA algorithm, we can designate a single sensor node as the reference node $\mathbf{x}_r$ and consider only the TDOA between this node and the remaining nodes. We can modify the problem formulation by rewriting (5) into

$$\Delta_{i,r} + \frac{1}{c}\|\mathbf{x}_r - \mathbf{y}\| = \frac{1}{c}\|\mathbf{x}_i - \mathbf{y}\| + n_i - n_r, \tag{10}$$

which leads to

$$\begin{aligned}\left(\Delta_{i,r} + \frac{1}{c}\|\mathbf{x}_r - \mathbf{y}\|\right)^2 - \frac{1}{c^2}\|\mathbf{x}_i - \mathbf{y}\|^2 \\ = \underbrace{\left(\frac{2}{c}\|\mathbf{x}_i - \mathbf{y}\| + n_i - n_r\right)(n_i - n_r)}_{\text{noise } w_i}.\end{aligned} \tag{11}$$

For $i = 1, \ldots, r-1, r+1, \ldots, N$. In the noise-free case, the right-hand side of (11) is of course zero. Thus, one approach to estimating $\mathbf{y}$ would be to minimize the maximum matching error between the squares of the propagation time between the ith sensor and the reference node. In [7], a modified LS method was used to find the global minimum solution to the sum of the matching errors in (11). The modified LS method needs to find the correction term to the LS inverse matrix, which is a search process involving matrix inverse and can be numerically ill-conditioned.

To find a globally convergent solution which is less sensitive to the noise correlation, we adopt the *min-max* criterion to obtain a simplified formulation

$$\tilde{\mathbf{y}} = \arg\min_{\mathbf{y}} \max_{i=1,\ldots,N}^{i\neq r} \left| \frac{1}{c^2}\|\mathbf{x}_i - \mathbf{y}\|^2 - \left(\frac{1}{c}\|\mathbf{x}_r - \mathbf{y}\| + \Delta_{i,r}\right)^2 \right|. \tag{12}$$

Although the min-max estimate remains nonconvex, it is quite amenable to semidefinite relaxations, as shown below. The resulting convex optimization algorithm turns out to be quite direct.

3.2 Outer-Product SDP Formulation of TDOA

We introduce one extra degree of freedom into the TDOA problem by setting $d_r = \|\mathbf{x}_r - \mathbf{y}\|$ as a variable in the variable vector $\bar{\mathbf{y}} = [\mathbf{y}^T \; d_r \; 1]^T$. We can then define the outer-product matrix

$$\mathbf{Y} = \bar{\mathbf{y}}\bar{\mathbf{y}}^T = \begin{bmatrix} \mathbf{y}\mathbf{y}^T & d_r\mathbf{y} & \mathbf{y} \\ d_r\mathbf{y}^T & d_r^2 & d_r \\ \mathbf{y}^T & d_r & 1 \end{bmatrix}$$

and

$$\chi_{ir} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{x}_i \\ \mathbf{0}^T & -1 & -d_{ir} \\ -\mathbf{x}_i^T & -d_{ir} & \mathbf{x}_i^T\mathbf{x}_i \end{bmatrix},$$

where $d_{ir} = c\Delta_{i,r}$ for $i = 1, \ldots, r-1, r+1, \ldots, N$. Using $\mathbf{Y}$ and χ_{ir}, (12) can be written in an equivalent form

$$\begin{aligned} &\min_{\bar{\mathbf{y}}, d_r} \; \tau \\ &\text{s.t. } -\tau < \frac{1}{c^2}\text{Trace}(\chi_i \mathbf{Y}) - \Delta_{i,r}^2 < \tau, \\ &i = 1, 2, \ldots, r-1, r+1, \ldots, N, \end{aligned}$$

which is a convex function in terms of variable $\mathbf{Y}$ and χ_i.

By applying semidefinite relaxation [11], we can formulate the problem into a convex one

$$\begin{aligned} &\min_{\mathbf{Y}, d_r} \; \tau \\ &\text{s.t. } -\tau < \frac{1}{c^2}\text{Trace}(\chi_{ir}\mathbf{Y}) - \Delta_{i,r}^2 < \tau, \\ &i = 1, 2, \ldots, r-1, r+1, \ldots, N, \\ &\mathbf{Y} \succeq 0, \\ &\mathbf{Y}(m+1, m+1) > 0, \\ &\mathbf{Y}(m+1, m+2) > 0, \\ &\mathbf{Y}(m+2, m+2) = 1. \end{aligned} \tag{13}$$

Note that $\mathbf{Y} \succeq 0$ denotes the (symmetric) positive semidefinite constraint.

The global optimal solution of this convex optimization can be found using modern SDP solvers such as SeDuMi [12] that applies the interior point method. Note that the solution of (13) gives $\mathbf{Y}$, which already contains the source location $\mathbf{y}$. We can directly get the source location from $\mathbf{Y}$. However, there is more information about $\mathbf{y}$ in the matrix $\mathbf{Y}$. To obtain a better estimation, we need to perform some postprocessing to convert the SDP relaxation solution $\mathbf{Y}$ into the original optimization problem in terms of $\tilde{\mathbf{y}}$. Some standard postprocessing techniques can be applied to extract $\tilde{\mathbf{y}}$ from $\mathbf{Y}$, as discussed in [13].

3.3 Inner-Product SDP Formulation of TDOA

As an alternative algorithm to the outer-product SDP formulation, we consider a different relaxation approach. We introduce a new variable for the inner-product $y_s = \mathbf{y}^T\mathbf{y}$, plus two additional variables $d_r = \|\mathbf{x}_r - \mathbf{y}\|, d_s = d_r^2$. Accordingly, the optimization problem (12) can be rewritten as

$$\begin{aligned} &\min_{\mathbf{y}, y_s, d_r, d_s} \; \tau \\ &\text{s.t. } -\tau < \frac{1}{c^2}\text{Trace}\left(\begin{bmatrix} \mathbf{I} & \mathbf{y} \\ \mathbf{y}^T & y_s \end{bmatrix}\begin{bmatrix} \mathbf{x}_i\mathbf{x}_i^T & -\mathbf{x}_i \\ -\mathbf{x}_i^T & 1 \end{bmatrix}\right) \\ &\quad - \frac{1}{c^2}\text{Trace}\left(\begin{bmatrix} 1 & d_r \\ d_r & d_s \end{bmatrix}\begin{bmatrix} d_{ir}^2 & d_{ir} \\ d_{ir} & 1 \end{bmatrix}\right) < \tau, \\ &i = 1, 2, \ldots, r-1, r+1, \ldots, N, \end{aligned} \tag{14}$$

where $d_{ir} = c\Delta_{i,r}$.

However, the two equalities $y_s = \mathbf{y}^T\mathbf{y}$ and $d_s = d_r^2$ are not affine. In order to make the whole formulation convex, we relax the two equalities $y_s = \mathbf{y}^T\mathbf{y}$ and $d_s = d_r^2$ to inequalities $y_s \succeq \mathbf{y}^T\mathbf{y}$ and $d_s \succeq d_r^2$, respectively. These inequalities can be expressed in linear matrix inequalities, i.e.,

$$\begin{bmatrix} \mathbf{I} & \mathbf{y} \\ \mathbf{y}^T & y_s \end{bmatrix} \succeq 0, \begin{bmatrix} 1 & d_r \\ d_r & d_s \end{bmatrix} \succeq 0. \quad (15)$$

Additionally, $d_s = d_r^2$ can be expressed in terms of $\mathbf{x}_r$, $\mathbf{y}$, and y_s, where $d_s = \mathbf{x}_r^T\mathbf{x}_r - 2\mathbf{x}_r^T\mathbf{y} + y_s$. Then, we can formulate the problem into

$$\begin{aligned} &\min_{\mathbf{y},y_s,d_r,d_s} \tau \\ \text{s.t. } &-\tau < \text{Trace}\left(\begin{bmatrix} \mathbf{I} & \mathbf{y} \\ \mathbf{y}^T & y_s \end{bmatrix}\begin{bmatrix} \mathbf{x}_i\mathbf{x}_i^T & -\mathbf{x}_i \\ -\mathbf{x}_i^T & 1 \end{bmatrix}\right) \\ &- \text{Trace}\left(\begin{bmatrix} 1 & d_r \\ d_r & d_s \end{bmatrix}\begin{bmatrix} d_{ir}^2 & d_{ir} \\ d_{ir} & 1 \end{bmatrix}\right) < \tau, \\ &i = 1, 2, \ldots, r-1, r+1, \ldots, N, \\ &\begin{bmatrix} \mathbf{I} & \mathbf{y} \\ \mathbf{y}^T & y_s \end{bmatrix} \succeq 0, \begin{bmatrix} 1 & d_r \\ d_r & d_s \end{bmatrix} \succeq 0, \\ &d_s = \mathbf{x}_r^T\mathbf{x}_r - 2\mathbf{x}_r^T\mathbf{y} + y_s. \end{aligned} \quad (16)$$

This optimization problem in (16) can now also be solved using standard (e.g., interior point) convex optimization methods such as SeDuMi [12].

We note that, unlike the outer-product relaxation, the inner-product relaxation can directly yield $\tilde{\mathbf{y}}$ without loss of information. Additional postprocessing is unnecessary for the inner-product relaxation. On the other hand, significant amount of location information would be lost from the result of the outer-product relaxation without postprocessing. Moreover, using techniques in [14], (16) can be shown to be equivalent to (13) without the constraint $d_r > 0$ and $d_s = \mathbf{x}_r^T\mathbf{x}_r - 2\mathbf{x}_r^T\mathbf{y} + y_s$. Since this constraint is consistent with the optimum solution, we expect this additional constraint to provide some performance improvement for the inner product algorithm.

3.4 Reference Node Selection

Although the CRLB of the TDOA model is invariant to the selection of reference node, no unbiased efficient estimators exist. Hence, the performance of practical localization algorithms varies with different reference nodes as we will show in Section 4. Therefore, we need to find a good reference node selection method.

We propose a very simple criterion for reference node selection here. Based on the principle that a more compact range of time difference of arrival $\Delta_{i,r}$ leads to better numerical conditioning, we select the reference node $\mathbf{x}_r$ based on the *median* criterion:

$$t_r = \text{median}\{t_i,\ i = 1,\ \ldots,\ N\}. \quad (17)$$

As will be shown by our simulation verifications, this simple reference selection criterion can generate good estimation.

TABLE 1
Complexity Comparison of Different Algorithms

Algorithm	Iteration Number	Operation Per Iteration
Classic	1	$O(mN)$
TSLS	1	$O((2m+1)N)$
SRDLS	1	$O((3m+5)N)$
YWL-ECR	$O((2m)^{1/2})$	$O((mN)^2)$
SDP-O	$O((m+2)^{1/2})$	$O((m+2)^2N)$
SDP-I	$O((m+1)^{1/2})$	$O((m+1)^2N)$

3.5 Complexity Analysis of the Proposed Algorithm

We apply the result of [11] to evaluate the complexity of various convex algorithms. We denote our outer-product and inner-product based algorithms as SDP-O and SDP-I, respectively, while labeling the algorithms in [3], [6], [7], [8] as Classic, TSLS, SRDLS, and YWL-ECR, respectively. In Table 1, we compare algorithm complexities in terms of the number of iterations and operations in each iteration. By comparison, we can see that the complexity of the YWL-ECR algorithm is considerably higher than other algorithms, particularly in terms of the operations in each iteration when the number of sensors N is large. On the other hand, all other methods have similar complexity because we have $m = 2$ or 3.

3.6 Further Estimation Refinement

In general, the SDP-I and SDP-O algorithms yield approximate solutions that are close to the optimum estimate. To further refine the estimate, it is customary to use the SDP estimate as the initial starting point of some traditional nonlinear search algorithms. For example, we can apply $\tilde{\mathbf{y}}$ as the starting point and (4) as the object function. We can then use the Powell minimum search algorithm [15] initialized with various SDP estimates to refine the estimation results. Note however, that additional search after applying various localization algorithm tends to obscure their performance differences. Thus, in the simulation results below, we will only compare the results of various localization algorithms without additional search.

4 Simulation Results

In our simulations, we test the proposed SDP-O and SDP-I algorithms along with other algorithms. Recall that the "classic (linear) algorithm" [3] obtains the location estimation through solving a set of linear equations. By adding a correction to the solution of the classic linear algorithm, the TSLS algorithm [6] improves the estimation. The SRDLS algorithm [7] also adds a correction term to the LS inverse matrix. The YWL-ECR algorithm [8] includes all the pairwise TDOA measurements and formulates the estimation problem into a convex problem via SDP relaxation. Our numerical simulation results are compared against the above algorithms. To be more fair, we also test the YWL-ECR algorithm by limiting to only one reference node and correlated TDOA noise, denoted as "YWL-ECR-SR." Moreover, we compare the results against the CRLB defined in (9).

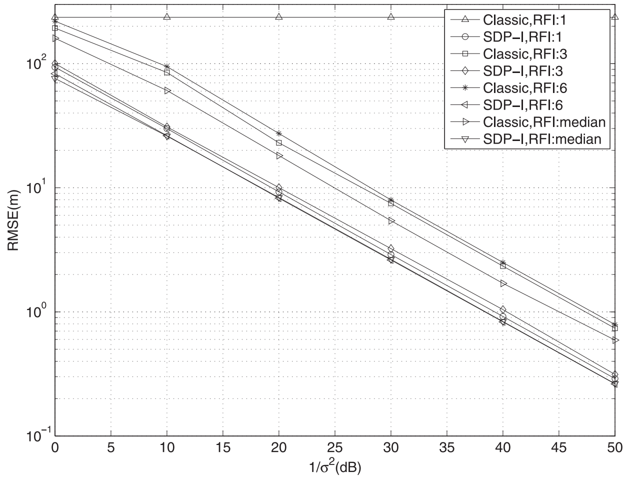


Fig. 1. Comparison of Classic and SDP-I algorithms using different reference nodes. RFI represents reference node index.

During the test, we place eight sensors in 2D at

$$\mathbf{x}_1 = [40, 40]^T, \mathbf{x}_2 = [40, -40]^T, \mathbf{x}_3 = [-40, 40]^T,$$
$$\mathbf{x}_4 = [-40, -40]^T, \mathbf{x}_5 = [40, 0]^T, \mathbf{x}_6 = [0, 40]^T,$$
$$\mathbf{x}_7 = [-40, 0]^T, \mathbf{x}_8 = [0, -40]^T.$$

In order to quickly determine a reference node, the *median* TOA measurement reference node selection criterion of (17) is applied. We compare the root mean squared error (RMSE) of the source position as standard deviation of the noise varies. The results are averaged over 3,000 Monte Carlo tests. For simplicity, we convert the noise in (2) to the distance domain. All the simulations are carried out on a PC with Intel Core 2 Duo P8400 CPU and 3 GB RAM running Matlab R7.6.0 (R2008a).

Example 1. In this example, we compare the performance of Classic and SDP-I by using different reference nodes. The source is placed at $\mathbf{y} = [200, 210]^T$, which is outside the convex hull formed by the sensor nodes. We use $\mathbf{x}_1$, $\mathbf{x}_3$, $\mathbf{x}_6$, and the node with *median* TOA measurement as the reference node, respectively. The performance comparison of Classic and SDP-I algorithms under different reference nodes is shown in Fig. 1 In particular, reference node selection based on the *median* criterion is tested in this example. We can see that the Classic method is more sensitive to the selection of reference node. When $\mathbf{x}_1$ is chosen as the reference node, it fails to locate the source even at very low noise. The difference between the SDP-I method under different reference nodes is very small. The results also illustrate that by selecting the reference node based on the simple *median* criterion, our localization algorithm SDP-I works well. Therefore, we will use this criterion in the next two examples.

Example 2. We place the source node at $\mathbf{y} = [20, 30]^T$, which is inside the convex hull of the sensor nodes. The estimation results of the Classic, SDP-O, SDP-I, TSLS, SRDLS, YWL-ECR-SR, and YWL-ECR are compared in Fig. 2. We set the parameter $\delta = 0.000001$ for YWL-ECR-SR and YWL-ECR. From the RMSE result, we can see that our proposed SDP-O and SDP-I are better than the Classic, SRDLS, and YWL-ECR-SR methods. YWL-ECR gives the best estimation in this case at the expense of the highest complexity by exploiting all pairwise TDOA information. We also illustrate the bias for the localization results. It is clear that none of the algorithms provides an unbiased estimate.

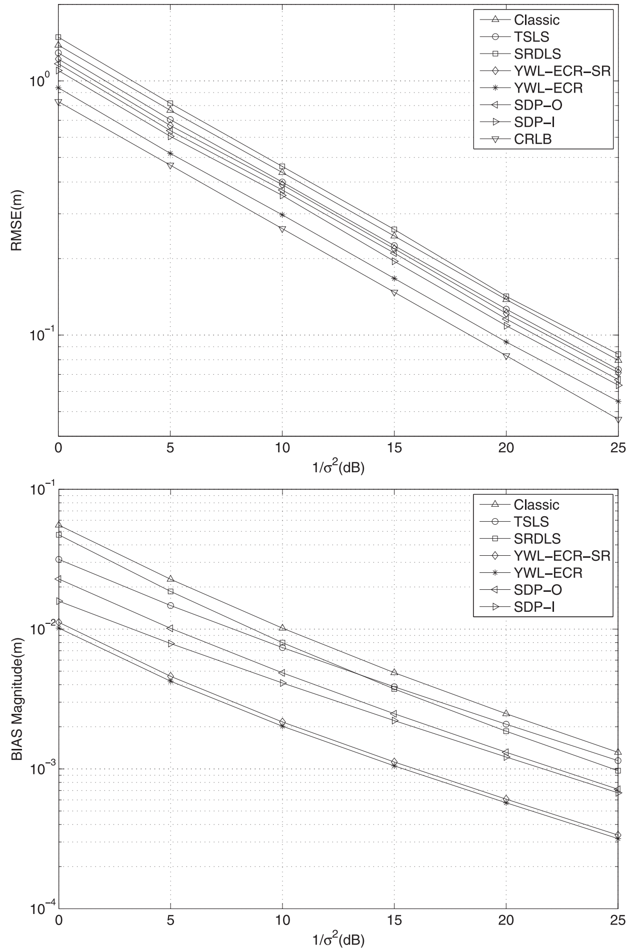


Fig. 2. Comparison of SDP-O, SDP-I, Classic, TSLS, SRDLS, YWL-ECR-SR, and YWL-ECR algorithms when a source node is inside the convex hull.

We give a comparison of the average CPU computational time perestimation trial in Table 2. We can see that the classic TDOA algorithm requires the least CPU time among all the algorithms while the YWL-ECR requires the most CPU time due to its highest complexity. The CPU time needs of our two proposed algorithms are moderate and fall in between the classic and the YWL-ECR algorithms, as expected.

Example 3. We place the source node at $\mathbf{y} = [10, 200]^T$, which is outside the convex hull formed by the sensor nodes. The estimation results of Classic, SDP-O, SDP-I,

TABLE 2
Average CPU Computational Time per Estimation for Different Algorithms

Algorithm	Average CPU Time (ms)
Classic	25.62
TSLS	61.17
SRDLS	140.82
YWL-ECR-SR	571.09
YWL-ECR	1358.69
SDP-O	358.75
SDP-I	263.57

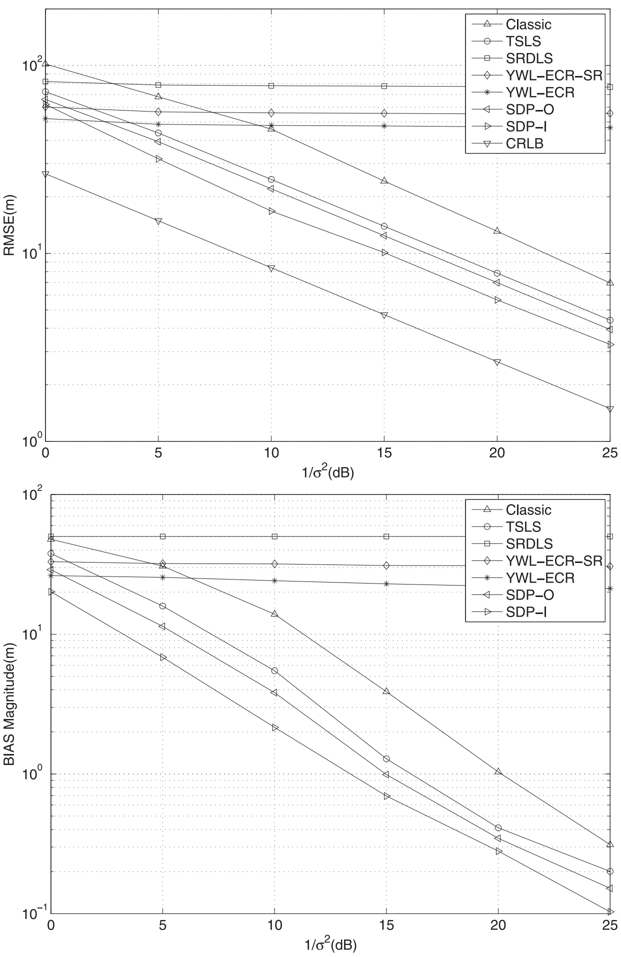


Fig. 3. Comparison of SDP-O, SDP-I, Classic, TSLS, SRDLS, YWL-ECR-SR, and YWL-ECR algorithms when a source node is outside the convex hull.

TSLS, SRDLS, YWL-ECR-SR, and YWL-ECR are shown in Fig. 3. We set $\delta = 0.000001$ for YWL-ECR-SR and YWL-ECR. It is interesting to observe that, unlike in the previous results, the SRDLS, YWL-ECR-SR, and YWL-ECR do not generate a good location estimate in this configuration. In fact, the RMSE fails to improve even as the noise variance is substantially reduced. For example, for the SRDLS algorithm, in a large percentage of Monte Carlo tests, an ill-conditioned matrix is inverted, leading to the poor numerical result. In fact, as shown in Fig. 3, the large estimate bias of the three algorithms contributed to the poor RMSE performance. This example illustrates the possible sensitivity of these existing methods previously unreported in the literature. By contrast, the proposed SDP-O and SDP-I algorithms both perform well in this case and are much better than the Classic and the TSLS. This result demonstrates the low sensitivity of the proposed algorithms to different sensor network configurations.

5 Conclusion

In this work, we propose an alternative convex optimization formulation for source localization in wireless sensor networks based on TDOA measurements. By designating a single reference node and using a min-max criterion that is less sensitive to measurement noise correlation, we present two SDP relaxation approaches that can efficiently solve the min-max problem. We also provide a simple and effective reference node selection method by choosing the node with the median TOA measurement. Complexity analysis and simulation results demonstrate that the proposed algorithms are effective and works well with low sensitivity to source-sensor configurations. Our proposed methods provide a good tradeoff between computational complexity and estimation performance.

Acknowledgments

The authors wish to thank the anonymous reviewers for several helpful suggestions and for providing [14]. The work is supported by the US National Science Foundation (NSF) Grants CCF-0830706, CCF-0803747, and CCF-0729025.

References

[1] E.Y. Xu, Z. Ding, and S. Dasgupta, "Robust and Low Complexity Source Localization in Wireless Sensor Networks Using Time Difference of Arrival Measurement," *Proc. IEEE Wireless Comm. and Networking Conf. (WCNC '10)*, Apr. 2010.

[2] N. Patwari, J.N. Ash, S. Kyperountas, A. Hero, R.L. Moses, and N.S. Correal, "Locating the Nodes: Cooperative Localization in Wireless Sensor Networks," *IEEE Signal Processing Magazine*, vol. 22, no. 4, pp. 54-69, July 2005.

[3] A.H. Sayed, A. Tarighat, and N. Khajehnouri, "Network-Based Wireless Location: Challenges Faced in Developing Techniques for Accurate Wireless Location Information," *IEEE Signal Processing Magazine*, vol. 22, no. 4, pp. 24-40, July 2005.

[4] P. Biswas, T.-C. Liang, K.-C. Toh, Y. Ye, and T.-C. Wang, "Semidefinite Programming Approaches for Sensor Network Localization with Noisy Distance Measurements," *IEEE Trans. Automation Science and Eng.*, vol. 3, no. 4, pp. 360-371, Oct. 2006.

[5] C. Meng, Z. Ding, and S. Dasgupta, "A Semidefinite Programming Approach to Source Localization in Wireless Sensor Networks," *IEEE Signal Processing Letter*, vol. 15, pp. 253-256, May 2008.

[6] Y.T. Chan and K.C. Ho, "A Simple and Efficient Estimator for Hyperbolic Location," *IEEE Trans. Signal Processing*, vol. 42, no. 8, pp. 1905-1915, Aug. 1994.

[7] A. Beck, P. Stoica, and J. Li, "Exact and Approximate Solutions of Source Localization Problems," *IEEE Trans. Signal Processing*, vol. 56, no. 5, pp. 1770-1778, May 2008.

[8] K. Yang, G. Wang, and Z. Luo, "Efficient Convex Relaxation Methods for Robust Target Localization by a Sensor Network Using Time Differences of Arrivals," *IEEE Trans. Signal Processing*, vol. 57, no. 7, pp. 2775-2784, July 2009.

[9] K.W.K. Lui, F.K.W. Chan, and H.C. So, "Semidefinite Programming Approach for Range-Difference Based Source Localization," *IEEE Trans. Signal Processing*, vol. 57, no. 4, pp. 1631-1633, Apr. 2009.

[10] A. Host-Madsen, "On the Existence of Efficient Estimators," *IEEE Trans. Signal Processing*, vol. 48, no. 11, pp. 3028-3031, Nov. 2000.

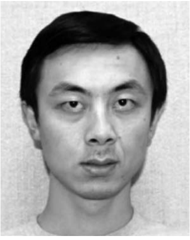
[11] L. Vandenberghe and S. Boyd, "Semidefinite Programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49-95, Mar. 1996.

[12] J.F. Sturm, "Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones," *Optimization Methods Software*, vols. 11/12, pp. 625-653, http://sedumi.mcmaster.ca, 1999.

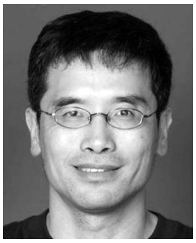
[13] Z. Ding and Z. Luo, "A Fast Linear Programming Algorithm for Blind Equalization," *IEEE Trans. Comm.*, vol. 48, no. 9, pp. 1432-1436, Sept. 2000.

[14] Y. Ding, D. Ge, and H. Wolkowicz, "On Equivalence of Semidefinite Relaxations for Quadratic Matrix Programming," Technical Report CORR 2010-02, Univ. of Waterloo, 2010.

[15] J. Nocedal and J. Stephen Wright, *Numerical Optimization.* Springer Verlag, 1999.



Enyang Xu received the BE and MS degrees from the National Mobile Communications Research Laboratory, Southeast University, Nanjing, China, in 2005 and 2008, respectively. Currently, he is working toward the PhD degree in the Department of Electrical and Computer Engineering at the University of California, Davis. His research interests include the area of wireless communications and signal processing, with current emphasis on source localization, tracking, joint detection, and decoding. He is a student member of the IEEE.



Zhi Ding received the PhD degree in electrical engineering from Cornell University in 1990. From 1990 to 1998, he was a faculty member of Auburn University. From 1998 to 2000, he was a faculty member of electrical and computer engineering at the University of Iowa. He joined the University of California, Davis in 2000 as a professor of electrical and computer engineering. Currently, he is working as a child family endowed professor of engineering and entrepreneurship at the University of California, Davis. He is also a guest Changjiang chair professor of the Southeast University in Nanjing, China. He has published more than 200 refereed research papers. He has also coauthored two books: *Blind Equalization and Identification* (Taylor and Francis, 2001) and *Modern Digital and Analog Communication Systems* (Oxford University Press, 2009). His major research interests include wireless communications, networking, and signal processing. He is a fellow of the IEEE and was named as a distinguished lecturer by the Circuits and Systems Society from 2004-2006. He was an associate editor for the *IEEE Transactions on Signal Processing* from 1994 to 1997 and 2001 to 2004. He served as an associate editor for *IEEE Signal Processing Letters* from 2002 to 2005. He was a member of the editorial board of the *IEEE Signal Processing Magazine*. He served as the technical program chair for COMSOC's Globecom 2006. He was a member of the IEEE Signal Processing Society's Technical Committee on Statistical Signal and Array Processing (1994-1998) and a member of the Technical Committee on Signal Processing for Communications (1998-2003). He also served on the IEEE Signal Processing Society Technical Committee on Multimedia Signal Processing (2002-2005). He was a distinguished lecturer of the IEEE Communications Society for the term 2008-2009.

Soura Dasgupta received the BE degree in electrical engineering from the University of Queensland, Australia, in 1980, and the PhD degree in systems engineering from the Australian National University in 1985. Currently, he is working as a professor of electrical and computer engineering at the University of Iowa. In 1981, he was a junior research fellow in the electronics and communications sciences unit at the Indian Statistical Institute, Kolkata. He has held visiting appointments at the University of Notre Dame, University of Iowa, Université Catholique de Louvain-La-Neuve, Belgium, and the Australian National University. His research interests include controls, signal processing, and communications. He served as an associate editor of the *IEEE Transactions on Automatic Control* from 1988 to 1991, as an editorial board member for the IEEE Control Systems Society Conference from 1998 to 2009, and as an editorial board member for the *IEEE Transactions on Circuits and Systems-II* from 2004 to 2007. He is a corecipient of the Gullimen Cauer Award for the best paper published in the *IEEE Transactions on Circuits and Systems* in 1990 and 1991, a past presidential faculty fellow, a subject editor for the *International Journal of Adaptive Control and Signal Processing*, and a member of the editorial board of the *EURASIP Journal of Wireless Communications*. He is a fellow of the IEEE.